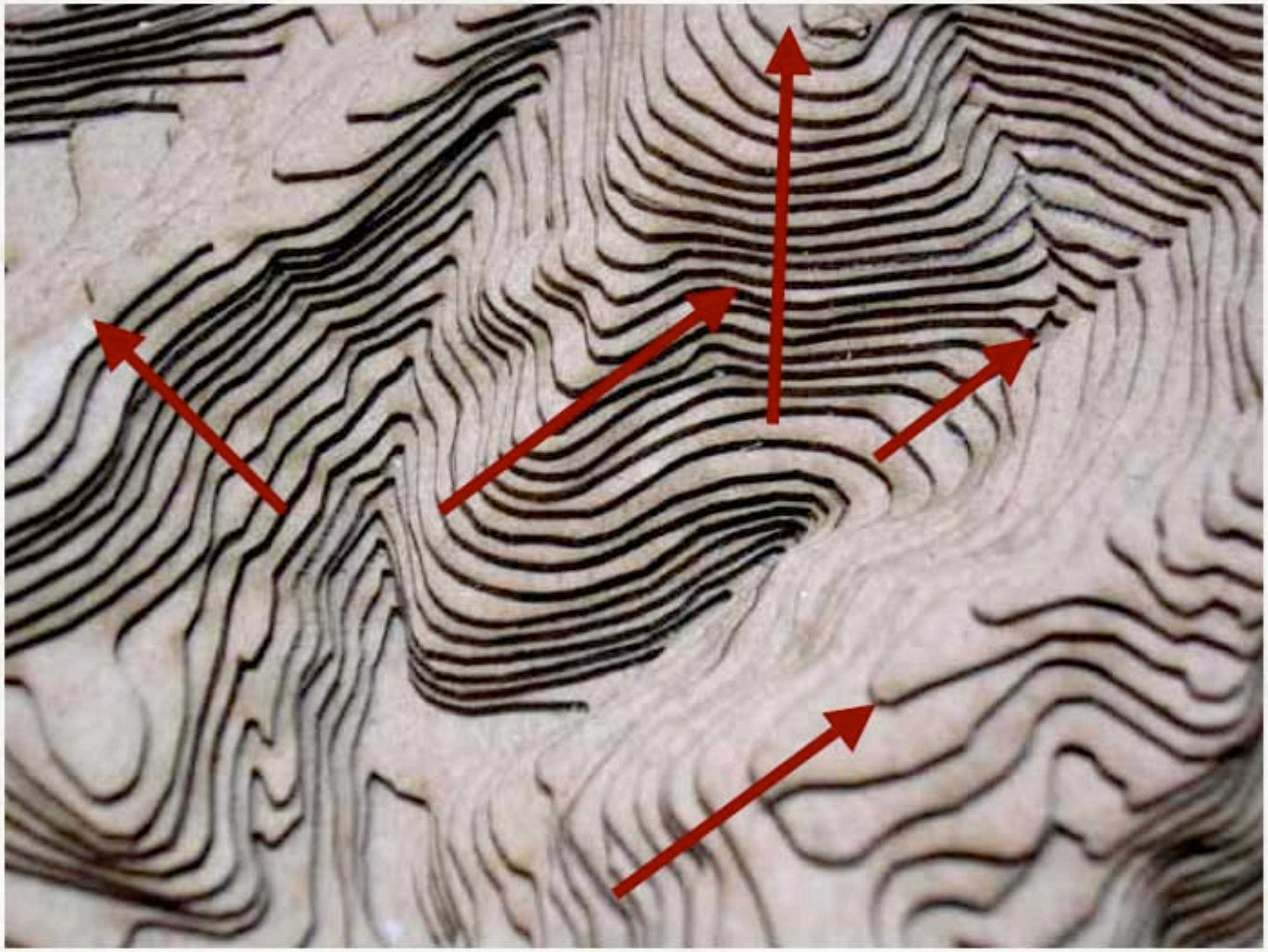
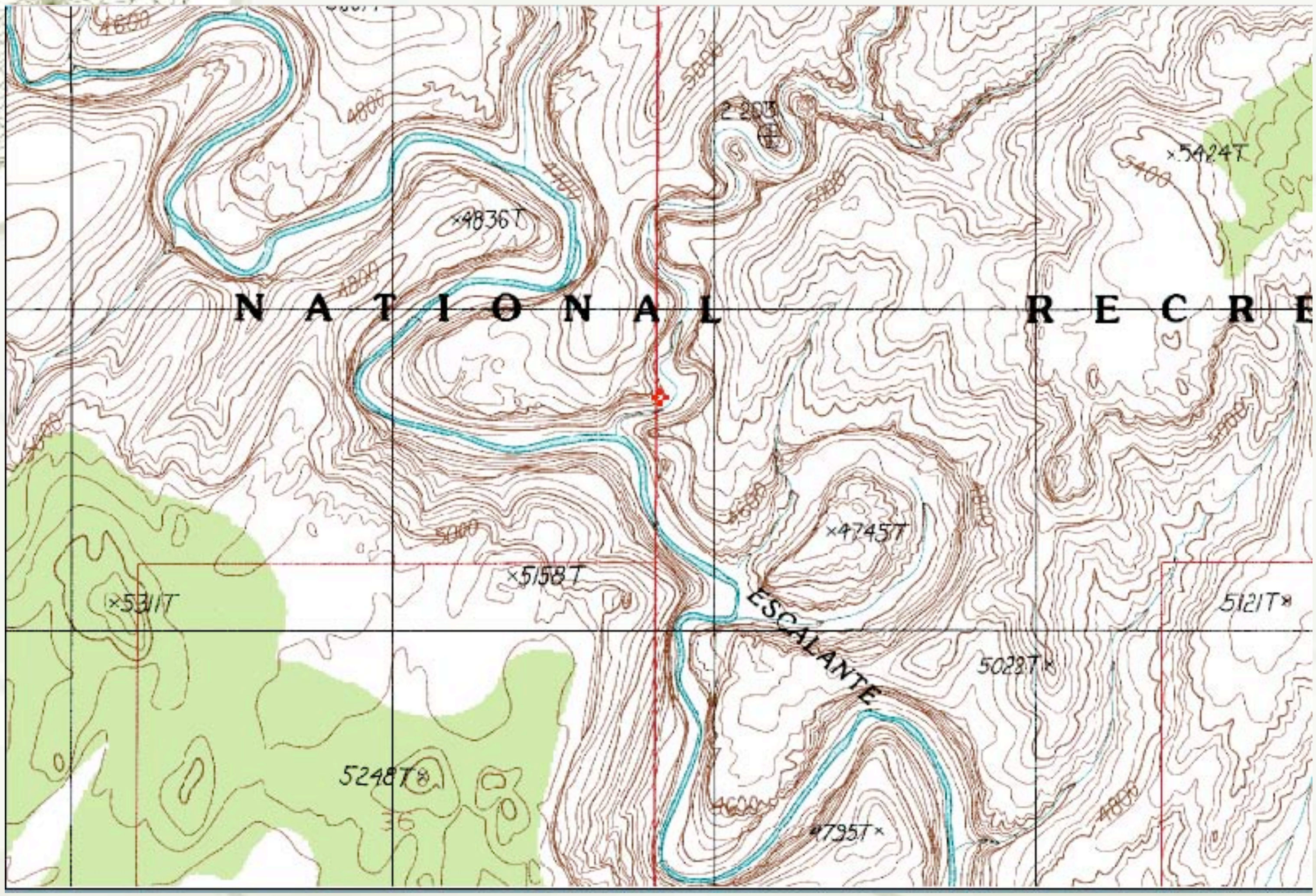
Which way is up?

Point uphill

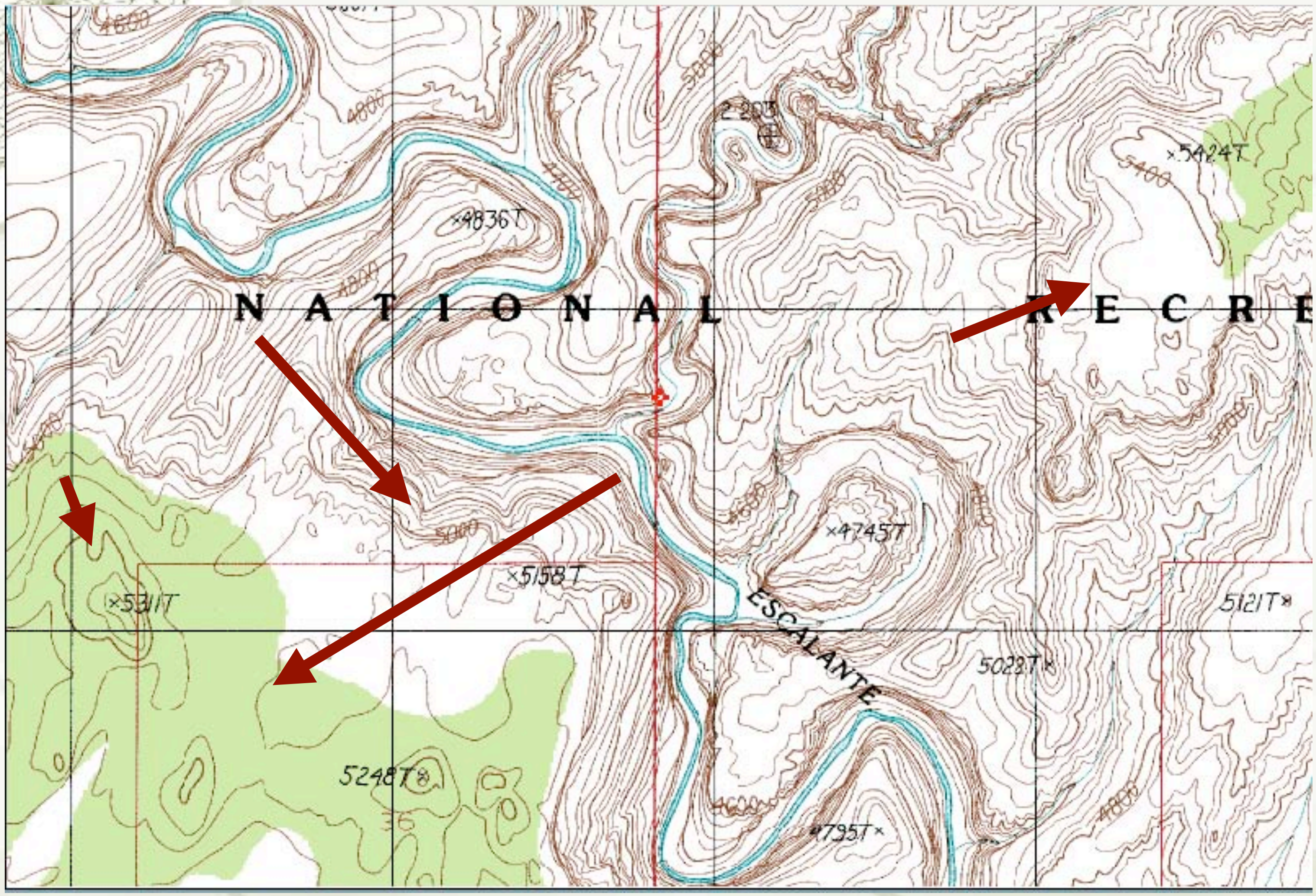
Point uphill



Point uphill



Point uphill



“Uphill” as a vector

- Direction
- Magnitude = the slope in that direction, e.g., 5% grade.

“Uphill” as a vector

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- Magnitude = the slope in that direction, e.g., 5% grade.
- If it happens that “uphill” is along the positive x-axis, then the slope is

$$\partial f / \partial x.$$

Directional derivative

- Choose a direction, by taking a unit vector **u**.

Directional derivative

- Choose a direction, by taking a unit vector **u**.
- Measure rise/run over a small distance in the direction **u**.

Directional derivative

- What is the d.d. if the vector **u** is *tangent* to a level curve?
- What is the d.d. if the vector **u** is *normal (perpendicular)* to a level curve?
- Of all directions **u**, which one gives the greatest d.d.?

Partial derivatives as a vector

- In 2 dimensions, we have 2 partials,

  $\partial F / \partial x$ and $\partial F / \partial y$

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Partial derivatives as a vector

- In 2 dimensions, we have 2 partials,

  $\partial F/\partial x$ and $\partial F/\partial y$
- In 3 dimensions we have 3

  $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$
- Same number as components of a vector, *hmmmmm*.

Differentiation as a *three-dimensional* concept

- Since you can't divide by a vector,

$$\frac{f(x+h) - f(x)}{h}$$

Differentiation as a *three-dimensional* concept

- Since you can't divide by a vector,
- We say f is differentiable iff

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{y} \bullet \mathbf{h} + o(\mathbf{h})$$

for some vector **y** (which will depend on **x**).

Differentiation as a *three-dimensional* concept

- Since you can't divide by a vector,
- We say f is differentiable at x = $\mathbf{x}_0$ iff

$$f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) = \mathbf{y} \bullet \mathbf{h} + o(\mathbf{h})$$

for some vector **y** (which will depend on $\mathbf{x}_0$).

- This is like the 1-D formula of a tangent line,

$$f(x_0+h) - f(x_0) = f'(x_0)\, h + o(h)$$

Differentiation as a *three-dimensional* concept

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  $f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) = \mathbf{y} \bullet \mathbf{h} + o(\mathbf{h})$,
- **y** is called the *gradient* of f at x_0, and denoted $\nabla f(\mathbf{x}_0)$.

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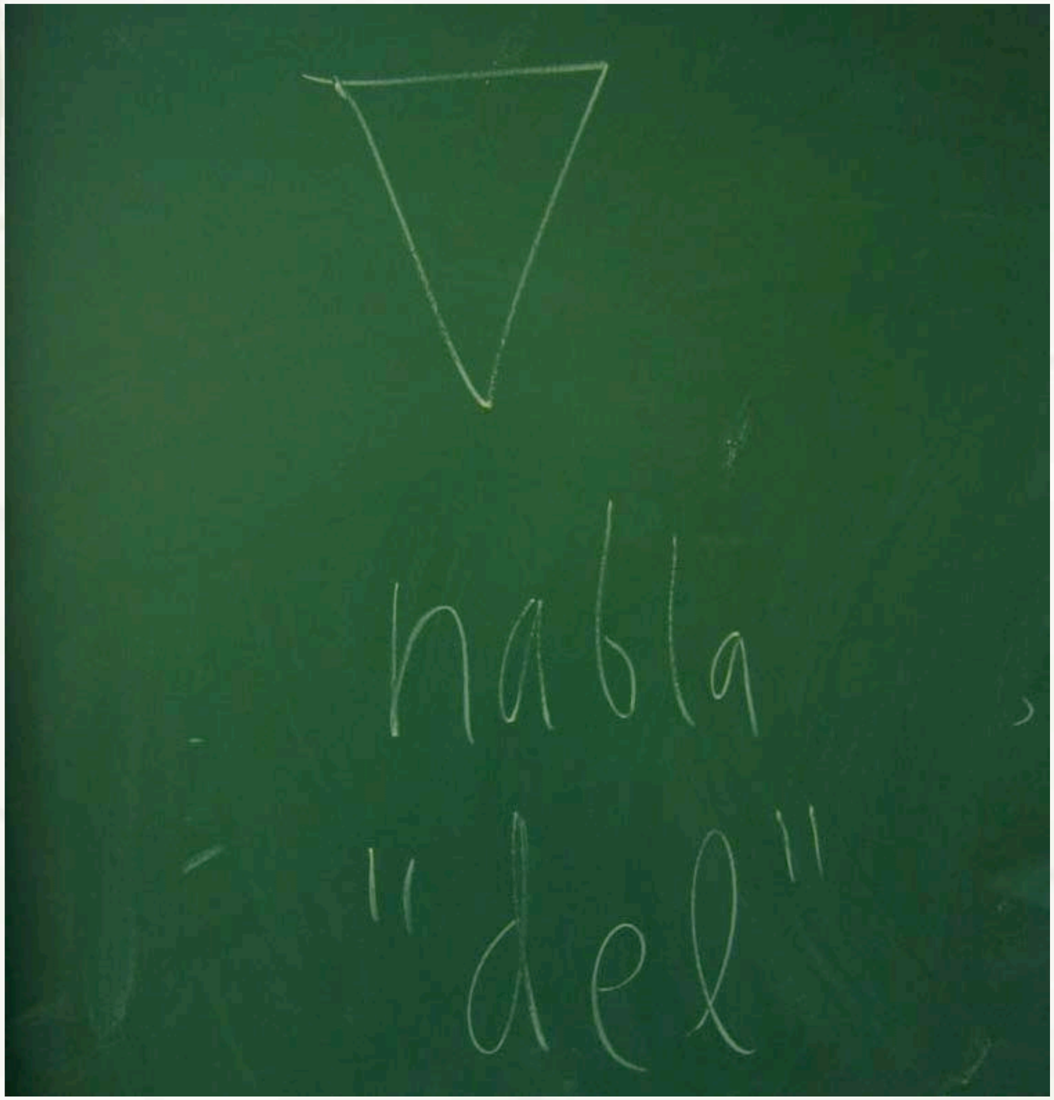
- *The gradient* $\nabla f(\mathbf{x})$ *of a scalar function is a vector-valued function of a vector variable!*


habla
"del"

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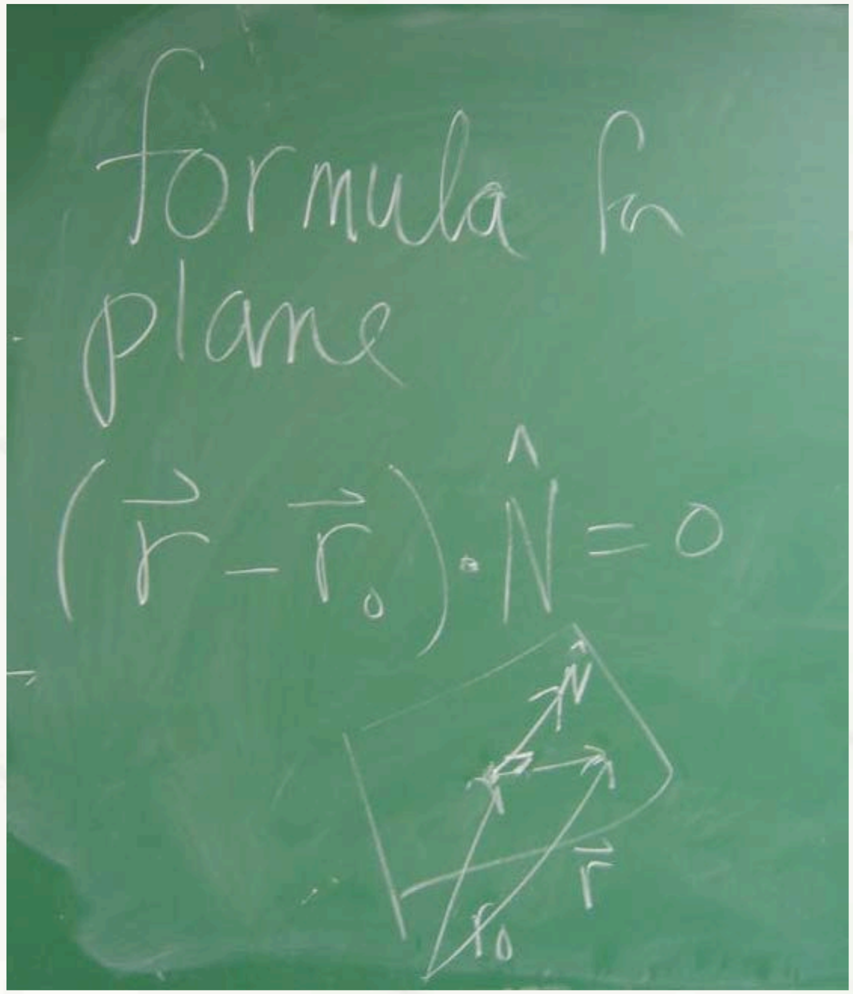
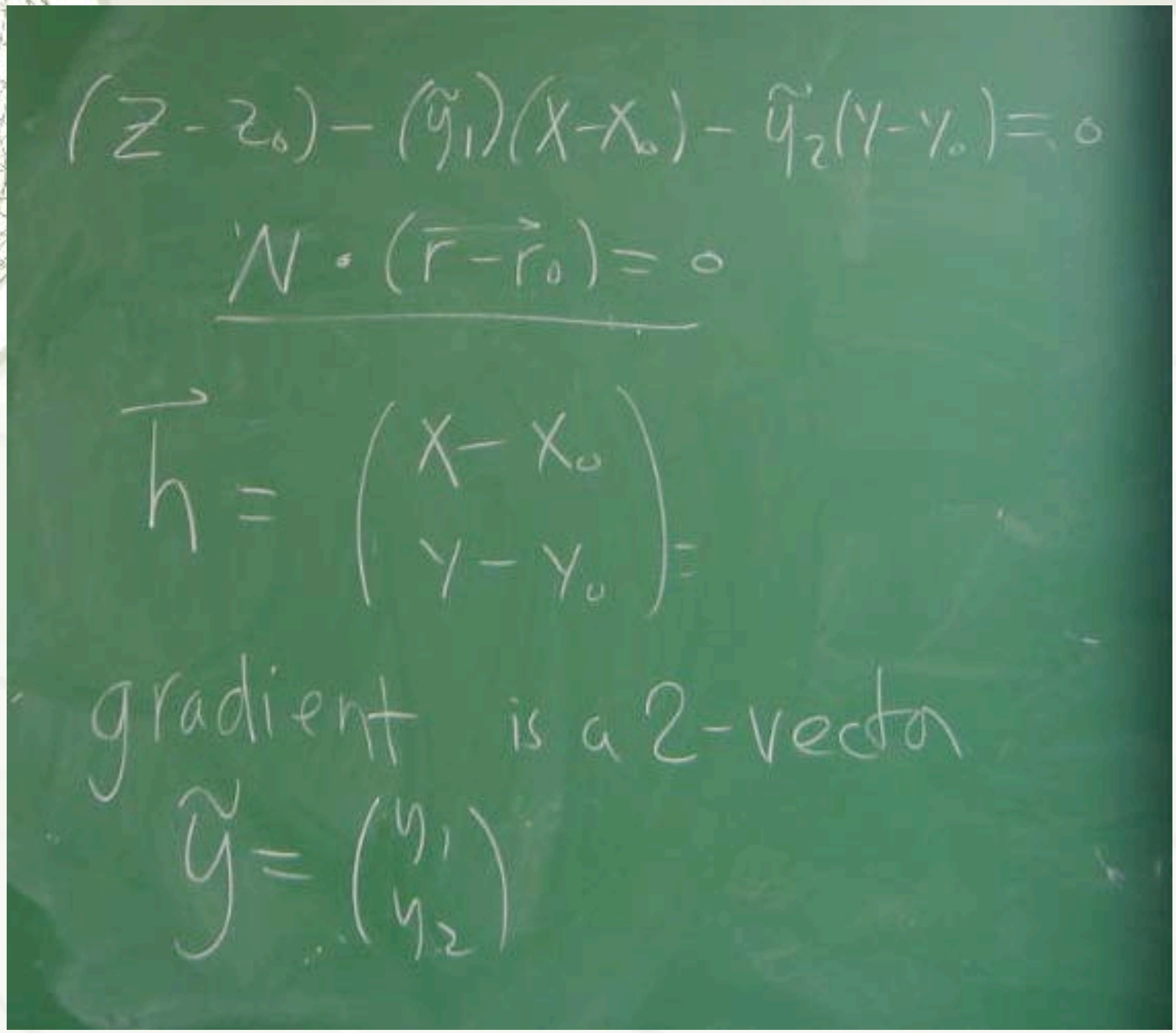
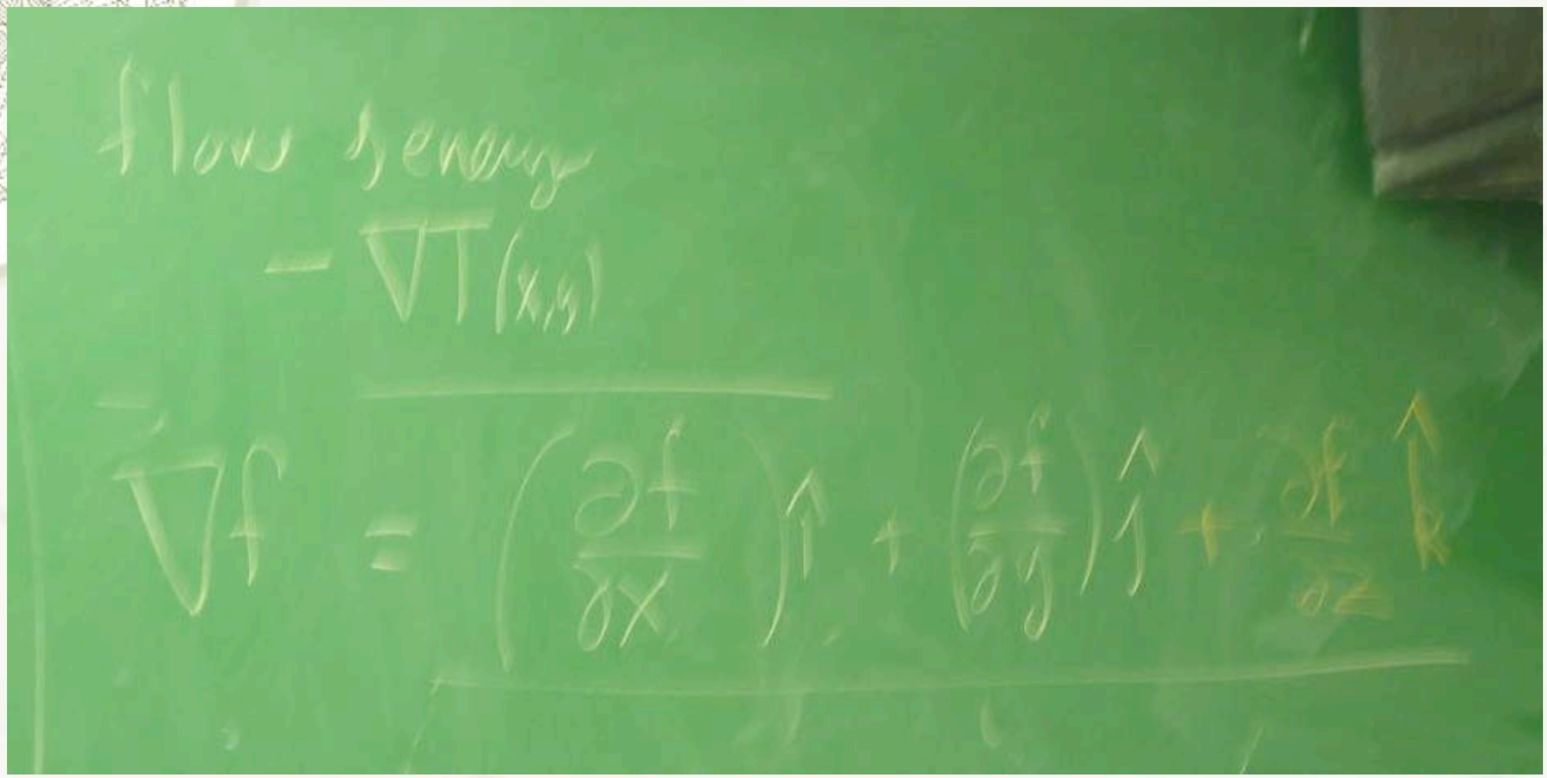
- It is easy to calculate the gradient.
- With the gradient you can easily calculate directional derivatives.
- “Uphill” is nothing other than the gradient!
- Tangent planes are also not hard to work out.

Some remarks mostly for the future, connecting the gradient to the notion of a "tangent plane."



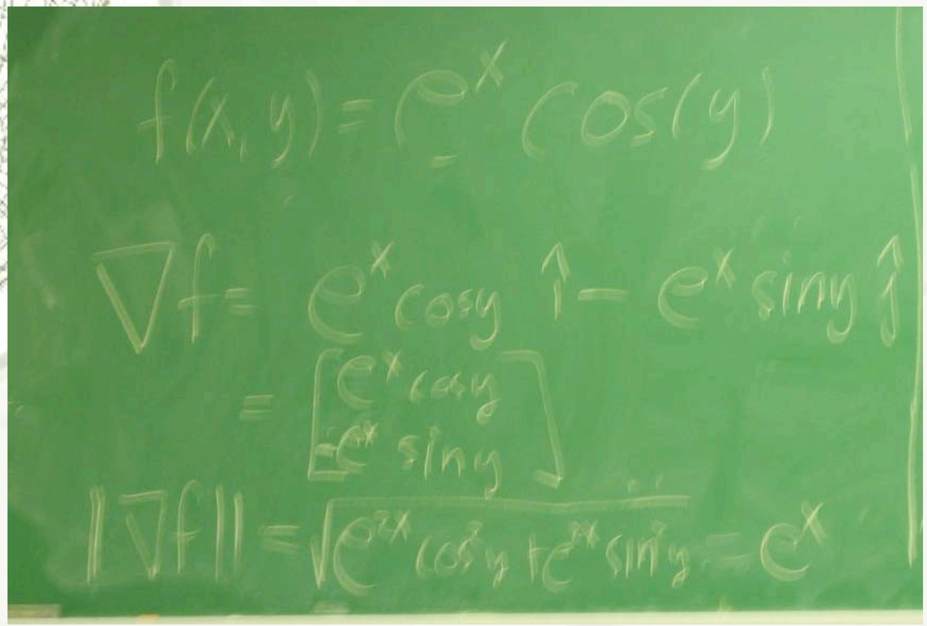

$(Z - Z_0) - (\tilde{y}_1)(X - X_0) - \tilde{y}_2(Y - Y_0) = 0$
$N \cdot (\vec{r} - \vec{r}_0) = 0$
$\vec{h} = \begin{pmatrix} X - X_0 \\ Y - Y_0 \end{pmatrix}$
gradient is a 2-vecta
$\tilde{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$



$-\nabla T(x,y)$
$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

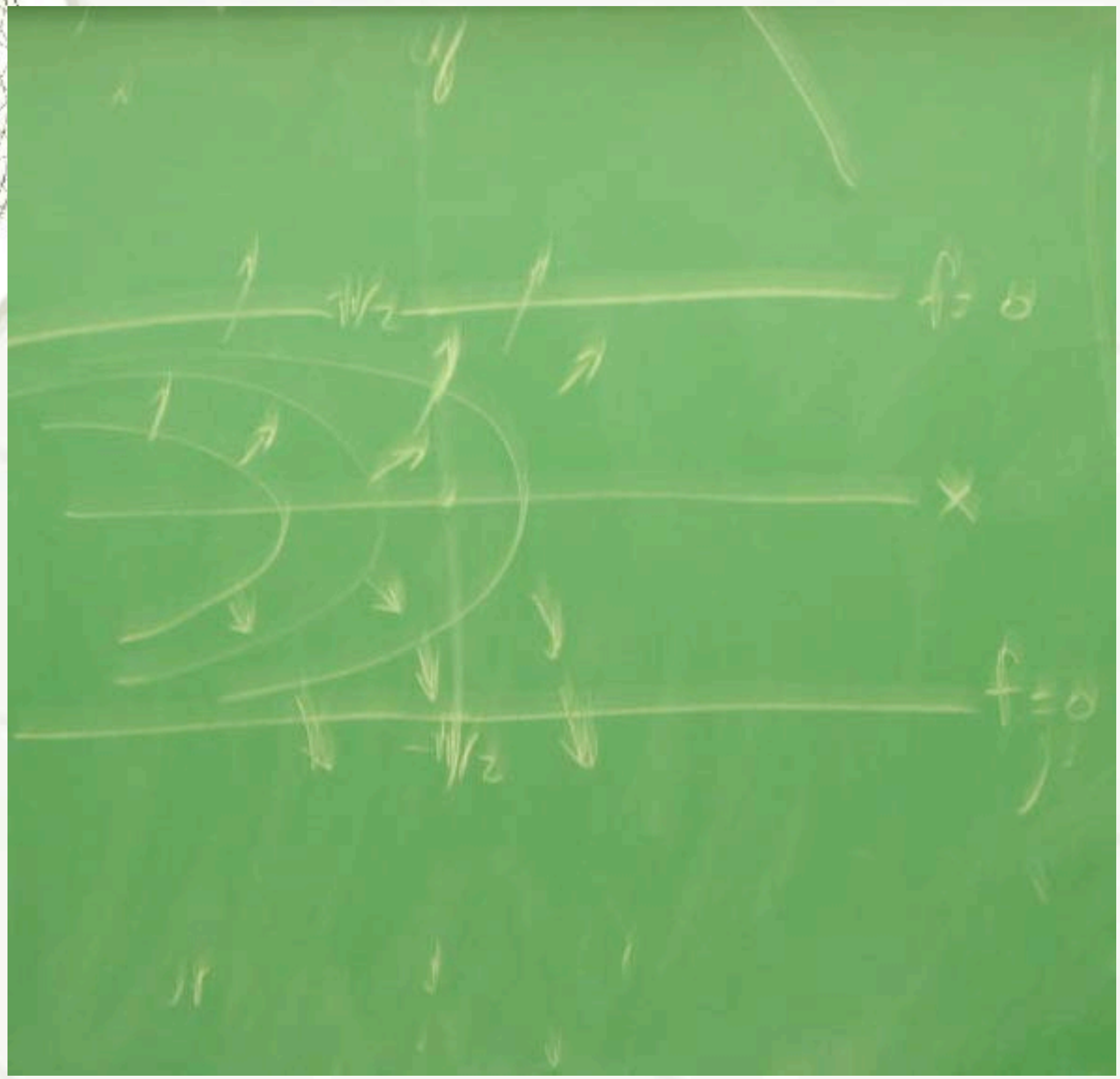
Now let's do some examples

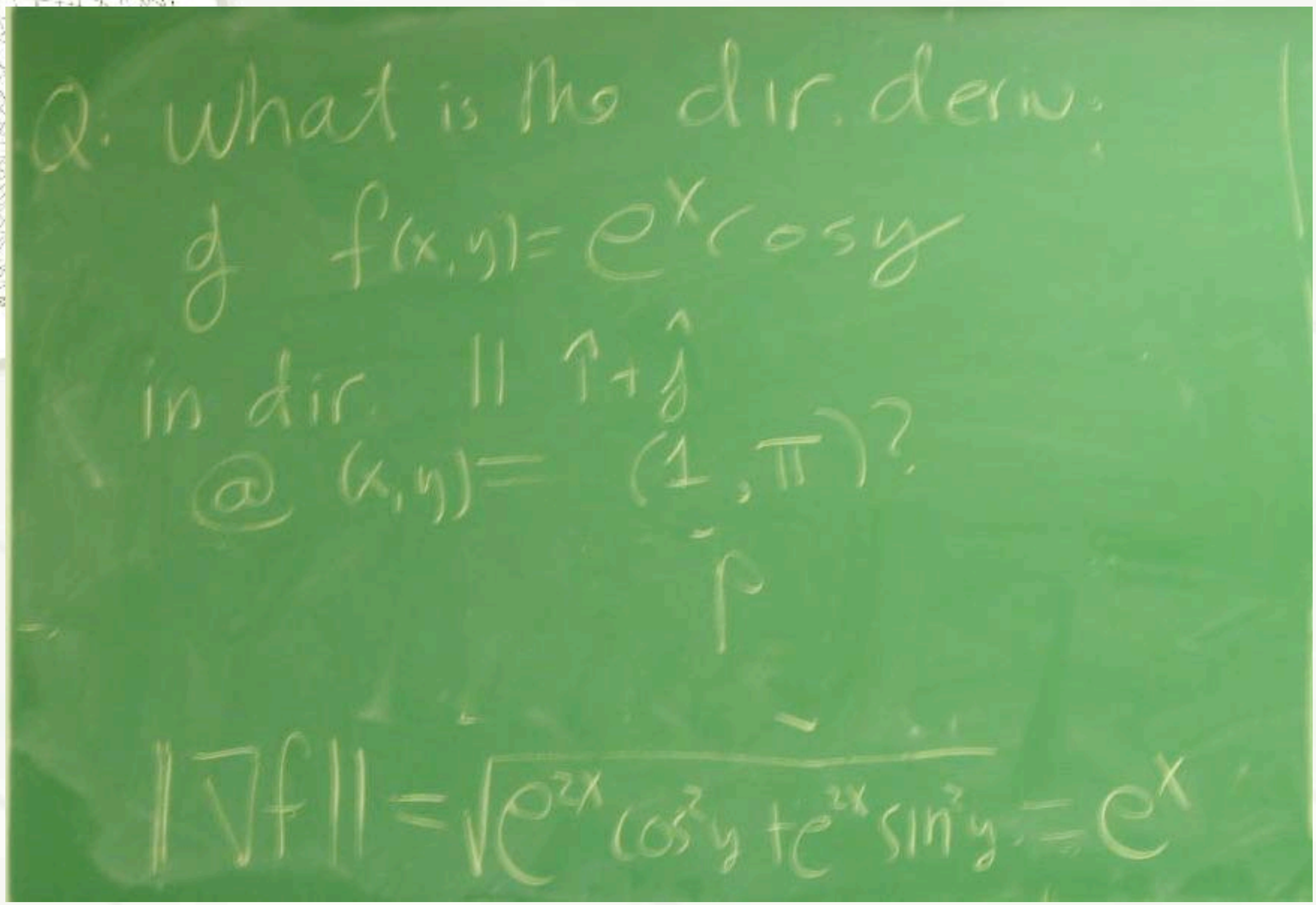
- Let's consider $f(x,y) = e^x \cos y$. (You might see this function in a thermodynamics class, since it is a possible equilibrium temperature distribution in a homogeneous solid.)



$\nabla f = e^x \cos y\ \mathbf{i} - e^x \sin y\ \mathbf{j}$

Magnitude is e^x.


f = 0
x
f = 0


Q: What is the dir. deriv.
of f(x,y)=e^x cos y
in dir. || î+ĵ
@ (x,y)= (1, π)?
||∇f|| = √(e^2x cos²y + e^2x sin²y) = e^x

1. Calculate the gradient
2. Identify a *unit* vector in the direction you want: $2^{-1/2}\,\mathbf{i} + 2^{-1/2}\,\mathbf{j}$.
3. Take dot product

ANSWER: $2^{-1/2} e^{1}(\cos \pi - \sin \pi) = -e/2^{1/2}$.

How to determine the gradient from a topographic map

- Direction - perpendicular to the "contour" (= level curve) passing through the position of interest.
- Calculate slope by measuring off a short distance and counting the contours crossed in that distance.
- Given a direction and a magnitude, you have the vector.

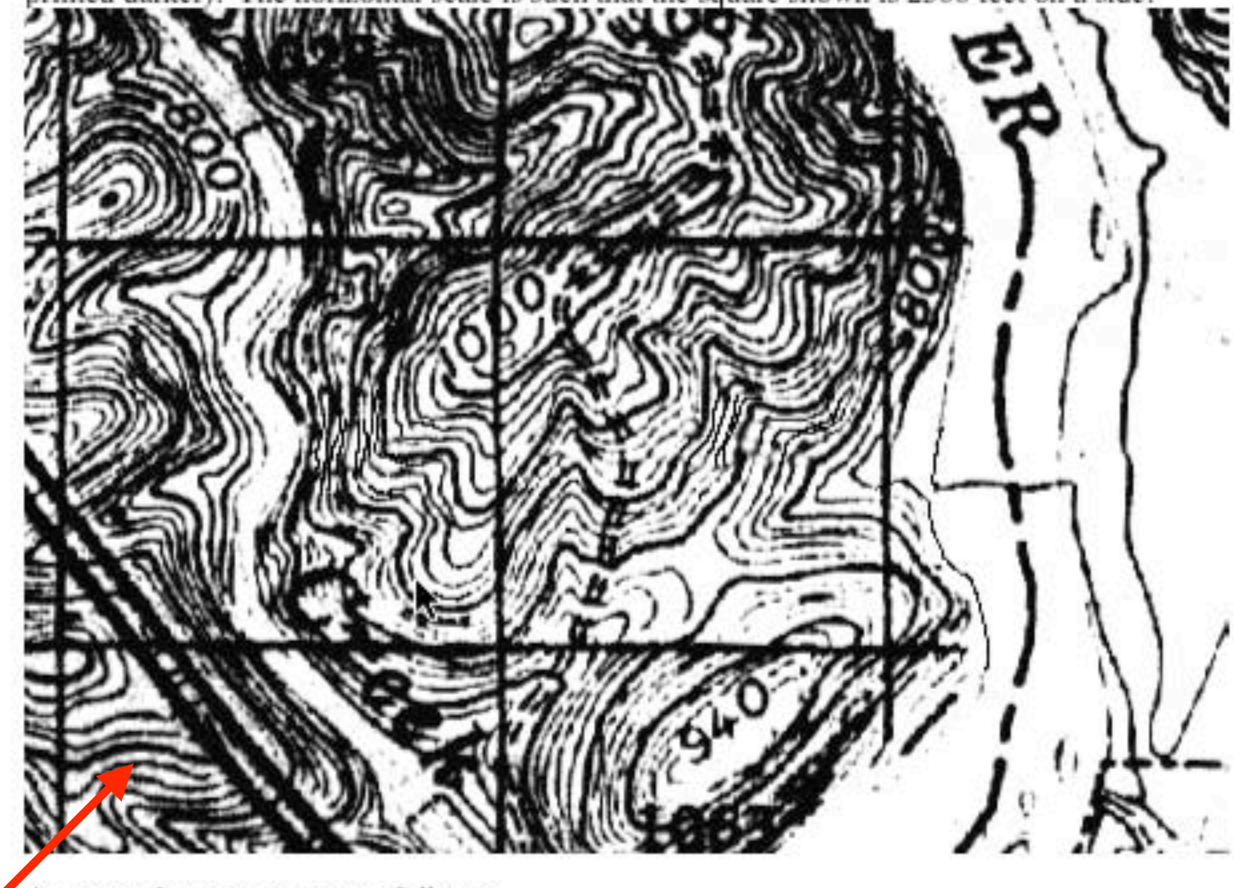
How to determine the gradient from a topographic map

- In the classroom example, from the Utah map, we found a spot where by moving uphill a distance 5/37 of a mile, we crossed 5 contours separated by 40 feet in height:
  - rise = 5*40 = 200
  - run = 5*5280/37
  - slope = .2803…

  A 28% grade is pretty darn steep, but far from the steepest on that map, I can tell you!

A problem from an old final exam:

A topographic map such as the one shown here, of the Chattahoochee National Recreation Area in Sandy Springs, is a contour plot for a function f(x,y). The contours are level sets for values of z=f(x,y). Contours are separated by heights of 10 feet (every fifth contour is printed darker). The horizontal scale is such that the square shown is 2500 feet on a side.



The point P

Annotate the contour map as follows:

a) Find the top of a hill and label it with the letter **T**.

b) Find a saddle point and label it with the letter **S**.

c) There are several cliffs in the park. Find a cliff on this map and label it with the letter **C**.

d) In the lower left part of the map you will find a small arrow. Estimate the gradient at the point of the arrow. By the way, the arrow drawn is not meant to indicate the gradient at its base.) Draw a vector on the map with the same direction as the gradient, and estimate the magnitude of the gradient here:________________