MATH 2411 - Harrell

The best and the brightest

and the worst and the dimmest

Lecture 10

Math 2411 – Honors Calculu × | Math 2411 Assignments ×

people.math.gatech.edu/~harrell/2411S13/HW2411x.html

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Mathematics 2411 **Honors Calculus III** **Spring, 2013**

Current reading and homework assignments

- Due Monday, 11 February

  NOTE: There will be a test on Wednesday, 13 February 

  Reading:
  - MT, Chapter 3 through Section 3.3
  - Lecture of 5 February
  - Lecture of 7 February

  Exercises (for test prep, not to be graded):
  - MT, Exercises 3,1, #7,8,14-16.
  - MT, Exercises 3.2, #3-9.
  - MT, Exercises 3.3, #1-5,14,17,19,25,28,29,38,41,42.
  - CHECK BACK LATER FOR MORE STUDY PROBLEMS

Current contests

Note about contest entries. These must be entirely your own work and not, for example, copied from the Web, even in modified form, which would be an honor code violation.

Unless otherwise specified, entries should be submitted to the professor, either in hard copy or by e-mail in a universally readab format, such as pdf.

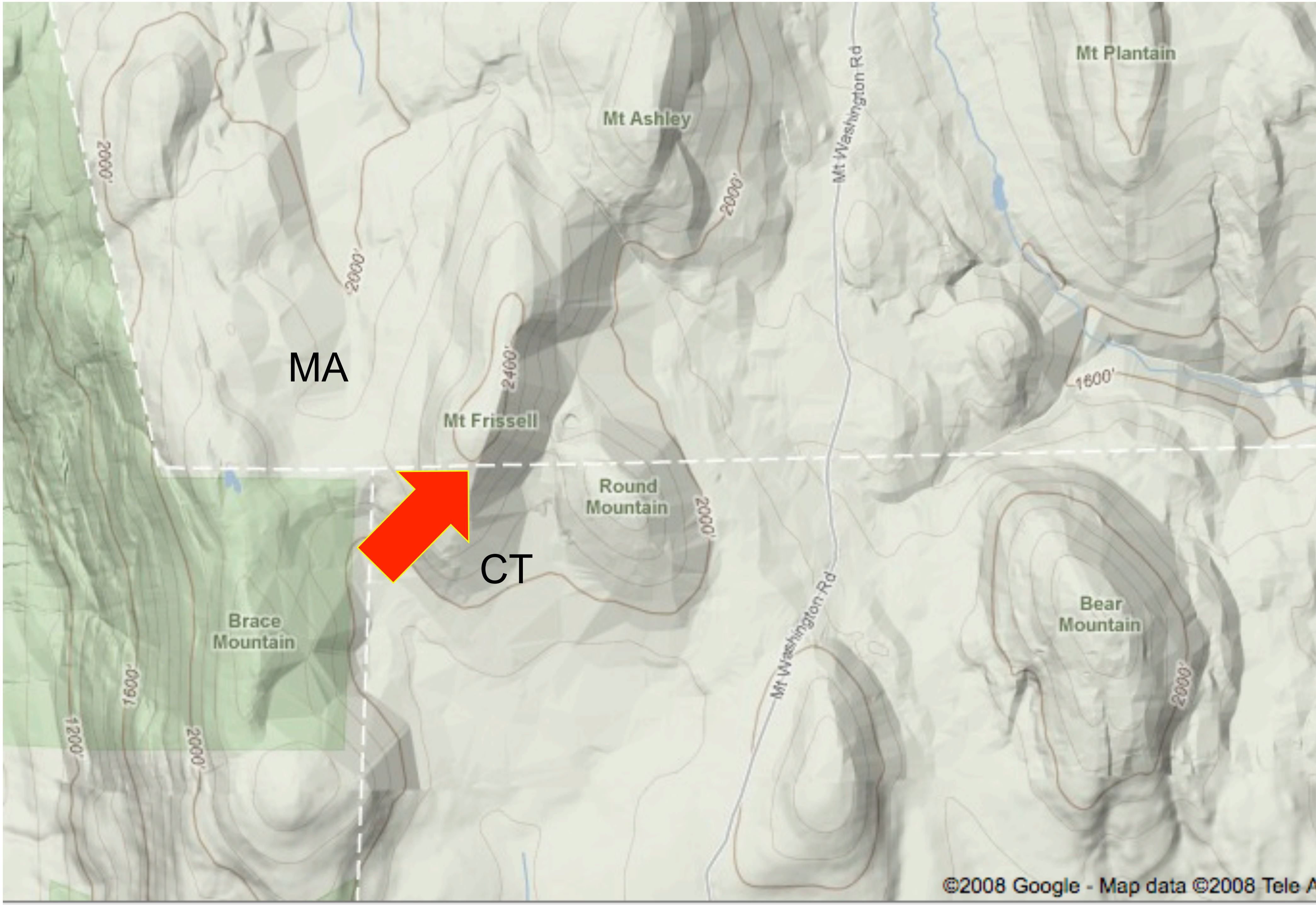
An important theorem

- A continuous function on a closed and bounded set *always* has a maximum value and a minimum value.

- Might or might not be true if the set is unbounded or not closed.

An important theorem

- A continuous function on a closed and bounded set *always* has a maximum value and a minimum value.
- Yet the gradient need *not* be **0** at the maximum.


Mt Plantain
Mt Ashley
Mt Washington Rd
2000'
2400'
1600'
MA
Mt Frissell
Round Mountain
CT
Brace Mountain
Bear Mountain
1200'
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Maxima and minima

- What is the maximum value of

  $$2 + 4x + xy - 3x^2 - 2y^2 \,?$$

  - How do I know there is a max even if there are no restrictions on the region, which ⇒ it is unbounded?

Maxima and minima

- What is the maximum value of

  $$2 + 4x + xy - 3x^2 - 2y^2 \,?$$

  $$\underbrace{\tfrac{1}{2}(2xy - x^2 - y^2)}_{\leq 0} - \underline{2\tfrac{1}{2}x^2 - 1\tfrac{1}{2}y^2}$$

  - How do I know there is a max even if there are no restrictions on the region, which ⇒ it is unbounded?

Maxima and minima

- What is the maximum value of

$f(\vec{r}) = 2 + 4\,x + x\,y - 3\,x^2 - 2\,y^2$?

$$\nabla f = \begin{pmatrix} 4 + y - 6x \\ x - 4y \end{pmatrix} = \vec{0} \quad \text{where?}$$

$$x_0 = 4y_0 \Rightarrow \quad 4 + y_0 - 24 y_0 = 0$$

$$y_0 = \frac{4}{23}$$

$$x_0 = \frac{16}{23}$$

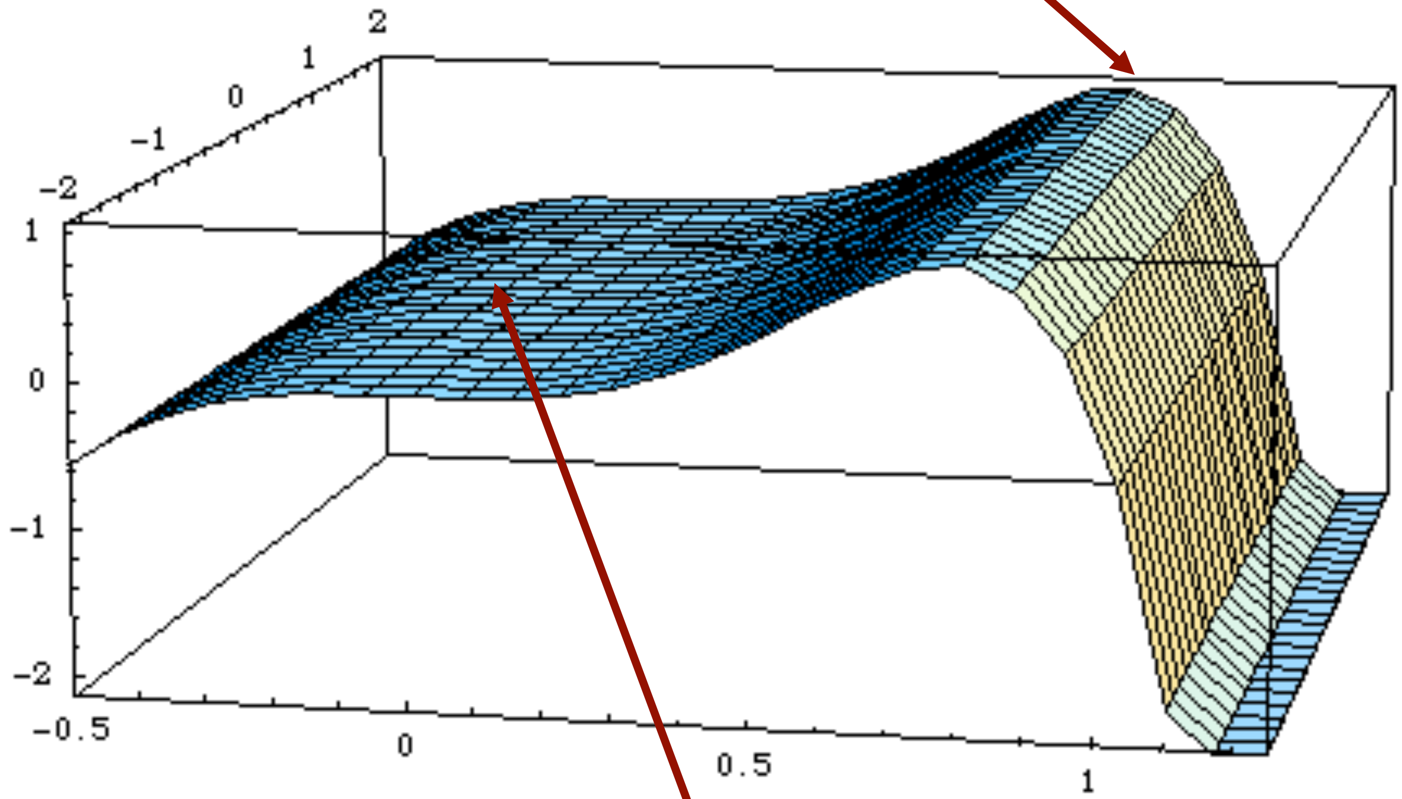
- Let U be an open, connected domain.
  - For *local* max/mins, assume open; it could be the *interior* of the closed domain of the main theorem.
- What is the condition for a local maximum or minimum?

Maxima and minima- Discussion

- Let U be an open, connected domain.
  - For *local* max/mins, assume open; it could be the *interior* of the closed domain of the main theorem.
- What is the condition for a local maximum or minimum?
  - It's still going to be a max if we vary x with y fixed, or vice versa. So the first partials must "vanish" (jargon for "= 0") or not exist. (crit pt)
  - That's *necessary*. Is it also sufficient?

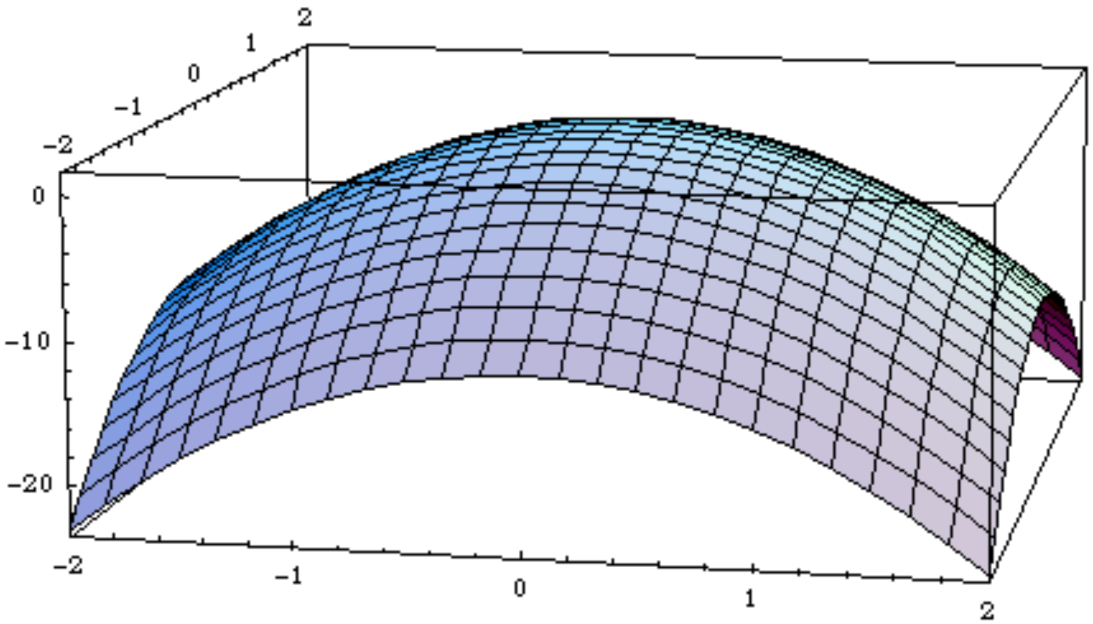
LOCAL MAX

```
In[12]:= Plot3D[4 x^3 - 4 x^6, {x, -1/2, 5/4}, {y, -2, 2},
  ViewPoint → {1, -5, 1}]
```



INFLECTION PT

```
In[4]:= Plot3D[1 - 3 x^2 - 3 y^2, {x, -2, 2}, {y, -2, 2}, ViewPoint → {1, -5, 1}]
```

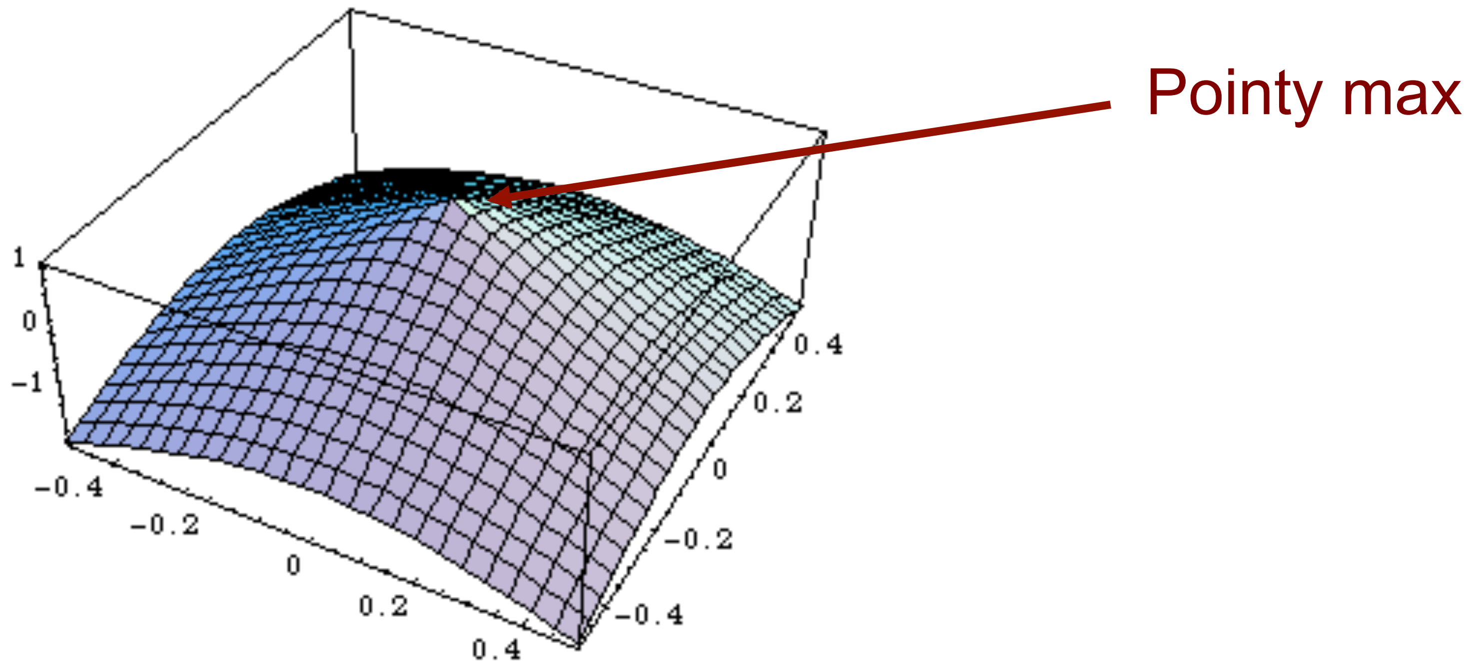


Example

- What is the maximum value of

$$1 - 4\,(x^2 + y^2)^{1/2} = 1 - 4\,|\mathbf{r}|\ ?$$

In[22]:= `Plot3D[1 - 4 Sqrt[x^2 + y^2], {x, -1/2, 1/2}, {y, -1/2, 1/2}]`



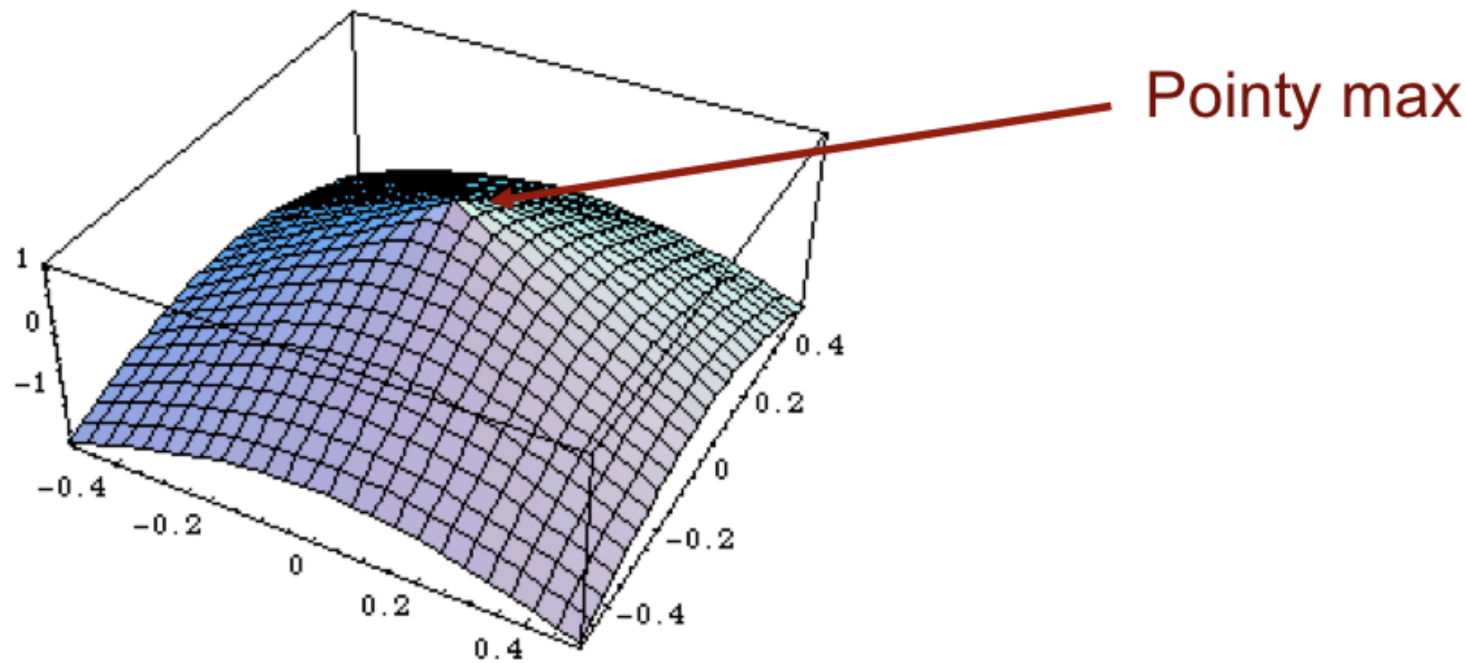
Example

$$\frac{\partial}{\partial x} \% = -4 \cdot \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}}$$

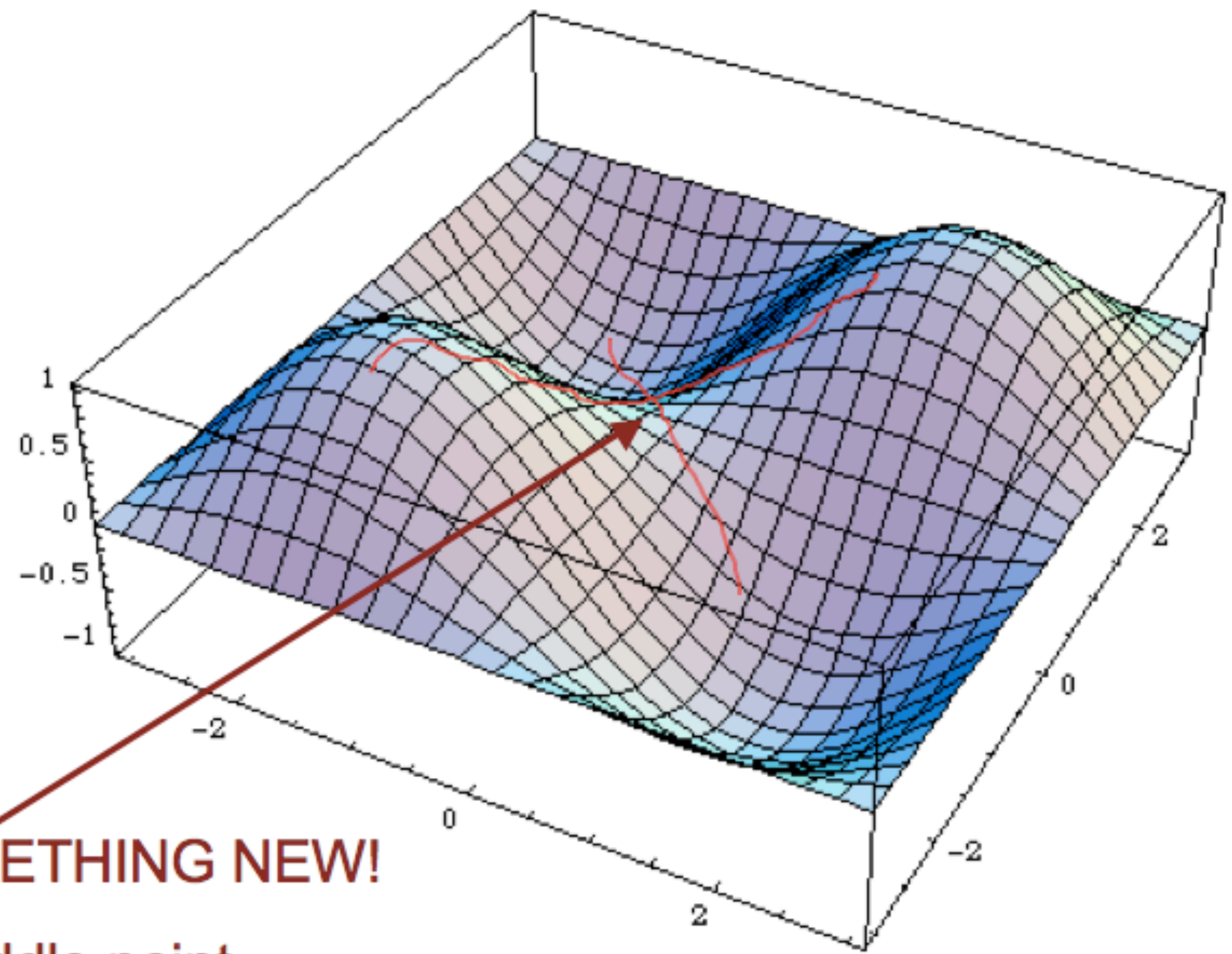
- What is the maximum value of

$$1 - 4\,(x^2 + y^2)^{1/2} = 1 - 4\,|\mathbf{r}|\ ?$$

```
In[22]:= Plot3D[1 - 4 Sqrt[x^2 + y^2], {x, -1/2, 1/2}, {y, -1/2, 1/2}]
```



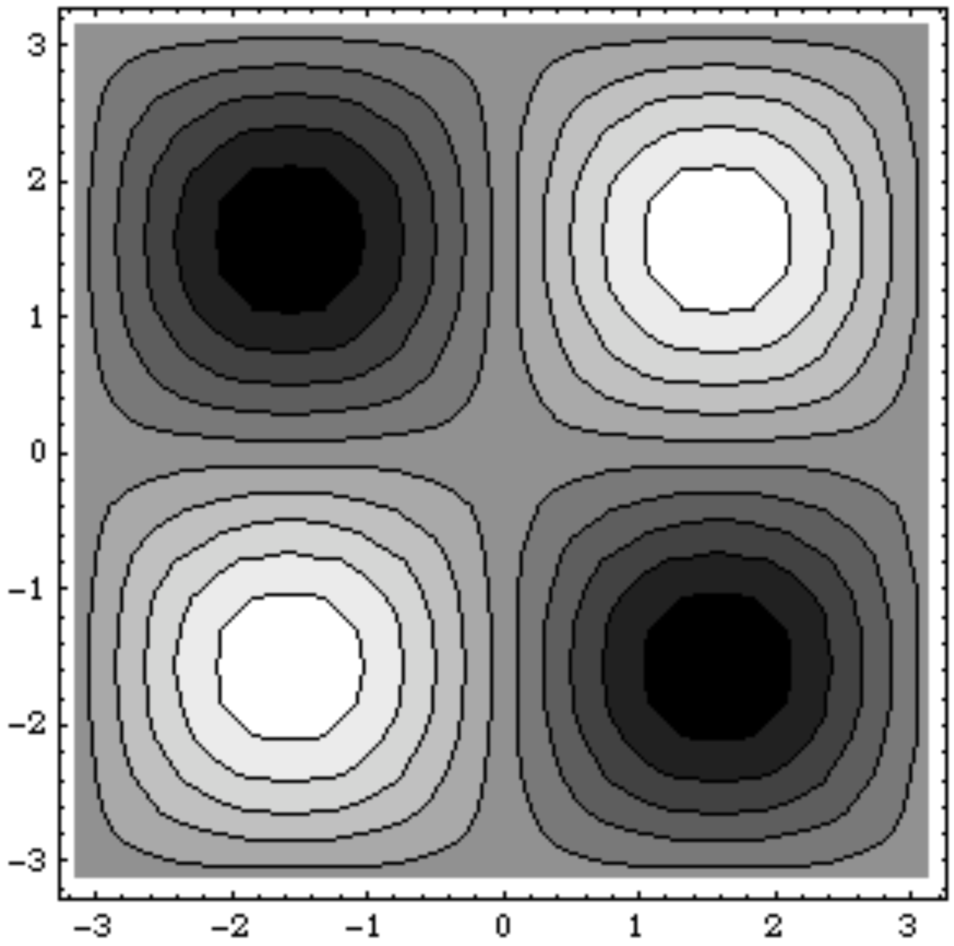
```
In[4]:= Plot3D[Sin[x] Sin[y], {x, -Pi, Pi}, {y, -Pi, Pi}]
```



SOMETHING NEW!

A saddle point

```
In[15]:= ContourPlot[Sin[x] Sin[y], {x, -Pi, Pi}, {y, -Pi, Pi}]
```



Example

- What is the maximum value of a 1D function like

  $$2 + 4x - 3x^2 ?$$

Second derivative test?

- Recall 1D: $f''(x) > 0 \Rightarrow$ local minimum,

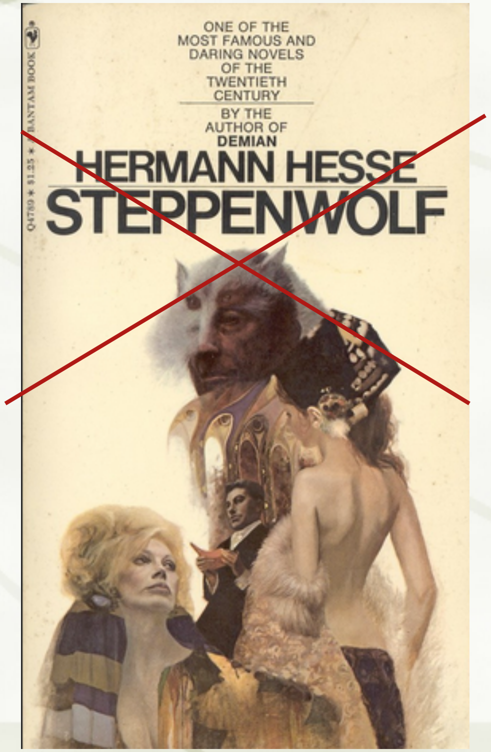
  $f''(x) < 0 \Rightarrow$ local maximum

Recall the 2nd order Taylor formula

$$f(\mathbf{x_0}+\mathbf{h}) = f(\mathbf{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x_0})h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x_0})h_i h_j + R_2(\mathbf{x_0},\mathbf{h}).$$

Second derivative test?

- We meet Dr. Hesse. No, not that Hesse!



Hermann Hesse

Second derivative test?

- The **Hessian** matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Ludwig Otto Hesse

Second derivative test?

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Ludwig Otto Hesse – the Determinator

Recall the 2nd order Taylor formula

$$f(\mathbf{x_0}+\mathbf{h}) = f(\mathbf{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x_0}) h_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x_0}) h_i h_j + R_2(\mathbf{x_0}, \mathbf{h}).$$

$\frac{1}{2} h^T H \vec{h}$

Second derivative test?

- The Hessian matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

A symmetric matrix may be positive definite, negative definite, or neither. What do those terms mean and how do you tell?

Second derivative test

- $D = \det(H) = f_{xx} f_{yy} - (f_{xy})^2$, "the discriminant."

  If at a crit. pt, and

  - $D < 0$, then SADDLE.
  - $D > 0$, then LOCAL MAX OR MIN. Check f_{xx} or f_{yy} to determine which:

    $f_{xx} > 0 \Rightarrow$ min, $\quad < 0 \Rightarrow$ max

Second derivative test?

- The Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Theorem: If det(H) < 0, then the critical point is a saddle.

Example

- sin(x) sin(y)   Where are the critical points, and what are they?

Recall the 2^{nd} order Taylor formula

$\nabla f \cdot \vec{h}$

$$f(\mathbf{x_0}+\mathbf{h}) = f(\mathbf{x_0}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x_0}) h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x_0}) h_i h_j + R_2(\mathbf{x_0}, \mathbf{h}).$$

$f(\vec{x}) = \sin(x)\sin y$

$\nabla f = \begin{pmatrix} \cos x \sin y \\ \sin x \cos y \end{pmatrix} \to \vec{0}$

near $\vec{x}_0 = (0,0)$

$\mathcal{H}[f] = \begin{pmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{pmatrix}$

$f(\vec{h}) = 0 + \vec{0} \cdot \vec{h} + \frac{1}{2} \vec{h} \cdot \mathcal{H} \vec{h}$

$+ \underline{\text{hots}}$

$= \frac{1}{2} \cdot 2xy + \text{hots}$

$(x,y) \to (0,0)$

$\mathcal{H} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Example

- What is the maximum value of

$$2 + 4x + xy - 3x^2 - 2y^2 \,?$$

$$\nabla f = \begin{pmatrix} 4 + y - 6x \\ x - 4y \end{pmatrix} \qquad d^2 f = \begin{pmatrix} -6 & 1 \\ 1 & -4 \end{pmatrix}$$

$$D = 23 > 0$$

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
- $x y e^{-2xy}$
- $x^3 + (x - y)^2$

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
  - (3, 0) is a local minimum (Hessian matrix has positives on diagonal and positive determinant.)
  - (-1,0) is a saddle. (Hessian matrix has negative determinant.)

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$

- $x\,y\,e^{-2xy}$

$$\nabla f = e^{-2xy}\begin{bmatrix} y - 2xy^2 \\ x - 2yx^2 \end{bmatrix}$$

$= 0$ where?

$x(1-2xy) = 0 \Rightarrow x = 0$ or $y = \frac{1}{2x}$

Case $x = 0$

1st cpt $\Rightarrow y = 0$

Case $y = 1/2x$

$0 = \frac{1}{2x} - \frac{\cancel{2x}}{(2x)^2}\cancel{x} \Rightarrow 0$

$$\mathcal{H}[f](x,y) = e^{-2xy}\begin{bmatrix} -2y^2-2y(y-2xy^2) & 1-4xy-2x(y-2xy^2) \\ 1-4xy-2xy(y-2xy^2) & -2x^2-2x(x-2yx^2) \end{bmatrix}$$

At $(x,y)=(0,0)$, we get $\mathcal{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow$ Saddle.

In the case where $y = 1/2x$, we simplify $\mathcal{H}$ by substituting:

$$\mathcal{H}[f](x,\tfrac{1}{2x}) = \frac{1}{e}\begin{bmatrix} -\frac{1}{2x^2} & -1 \\ -1 & -2x^2 \end{bmatrix} \Rightarrow D = \frac{1}{e^2}(1-1) = 0$$

It turns out that one eigenvalue is 0 and one < 0.

On our curve, xye^{-2xy} is a constant $\frac{1}{2e}$, and each pt. of the curve is a local max.