MATH 2411 - Harrell

新年快乐

Lecture 11

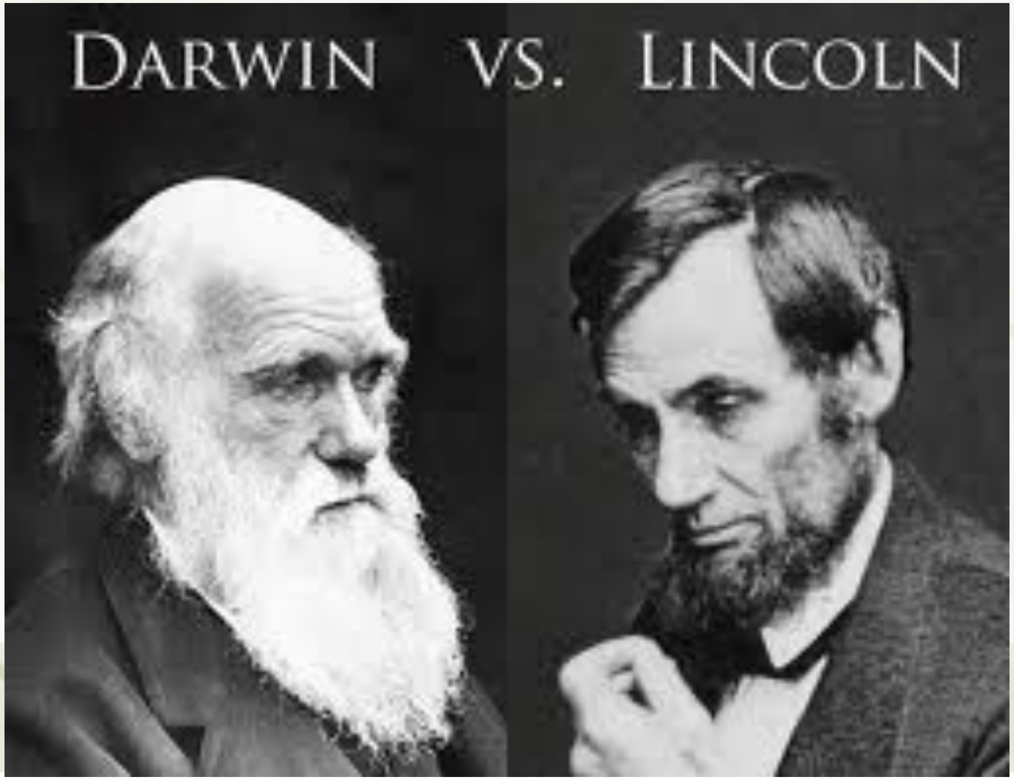
MATH 2411 - Harrell

It was the best of times, it was the worst of times

Lecture 11

This week's learning plan and announcements

- No office period next Monday
- Happy 204th birthday to Chuck and Abe!



This week's learning plan and announcements

- No office period next Monday
- Oh, and....

Math 2411 – Honors Calculu × Math 2411 Assignments ×

people.math.gatech.edu/~harrell/2411S13/HW2411x.html

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Mathematics 2411 **Honors Calculus III** **Spring, 2013**

Current reading and homework assignments

- Due Monday, 11 February

NOTE: There will be a test on Wednesday, 13 February 

Reading:
  - MT, Chapter 3 through Section 3.3
  - Lecture of 5 February
  - Lecture of 7 February
  - Lecture of 12 February
  - Chapters 7, 8, and 9 (especially the supplementary material), of CH.

Exercises (for test prep, not to be graded):
  - MT, Exercises 3,1, #7,8,14-16.
  - MT, Exercises 3.2, #3-9.
  - MT, Exercises 3.3, #1-5,14,17,19,25,28,29,38,41,42.
  - The exercises in 7, and 8, of CH could also be instructive.
  - Don't forget the bank of old tests.

Current contests

Note about contest entries. These must be entirely your own work and not, for example, copied from the Web, even in modified form, which would be a

Unless otherwise specified, entries should be submitted to the professor, either in hard copy or by e-mail in a universally readable format, such as pdf.

We'd better get some practice with:

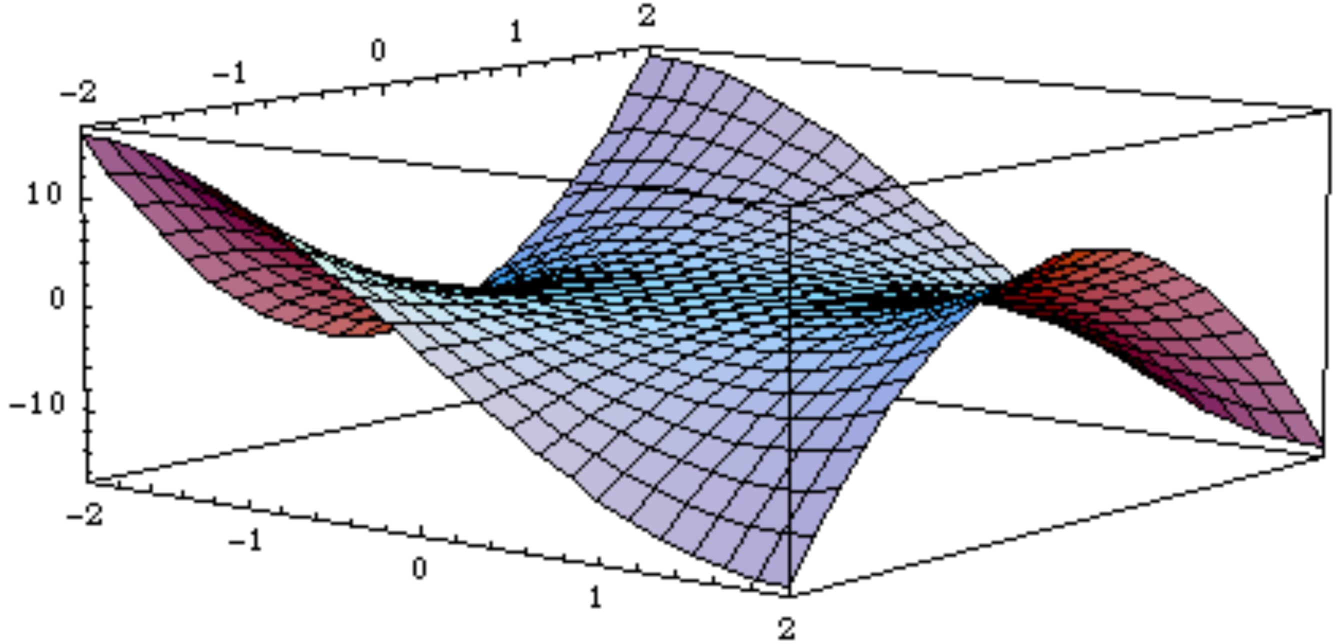
- What kind of surface is that?
- Closest/farthest points and other max/min problems
- Locating and classifying cp's
- Quick and dirty Taylor
- Normals, tan planes, normal lines
- Directional derivatives

Let's play...
"What kind of surface is that?"

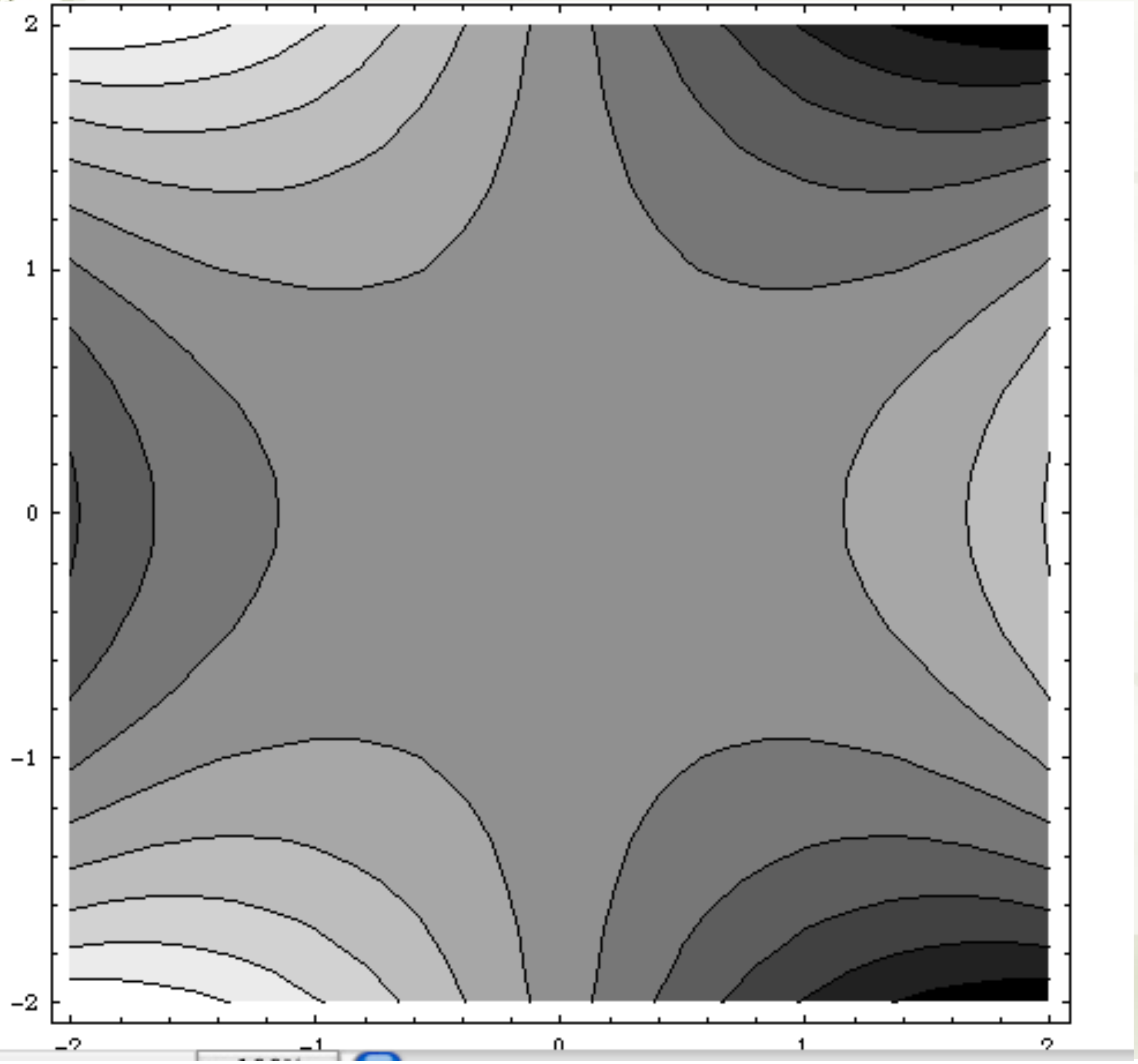
$$9x^2+4y^2+36z^2-36=0$$

Unchanged if (x,y,z)
$\longrightarrow (-x,y,z)$

A funky example



A funky example



Extra credit contest due Thursday

Due Thursday, 14 Feb. This contest has to do with the "funky example.” Find a formula for a saddle like that one. What do the gradient and Hessian matrix tell you about the funky saddle point at the origin $x = y = 0$? Carefully discuss the tangent planes and the (3D) normal vectors at points near the origin.

Closest and farthest points

1. How about a plane that misses the origin?

Closest and farthest points

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$A: \; 4x+3y+5z = 7$

b) $x^2+y^2+z^2$ $\quad \Rightarrow z = \frac{1}{5}(7-4x-3y)$

$f(x,y) = x^2+y^2 + \frac{1}{25}(49+16x^2+9y^2 - 56x - 42y + 24xy)$

$$0 = \nabla f = \begin{bmatrix} \frac{82}{25}x - \frac{56}{25} + \frac{24y}{25} \\ \frac{68}{25}y - \frac{42}{25} + \frac{24x}{25} \end{bmatrix}$$

Closest and farthest points

1. How about a plane that misses the origin?

Another way

$$\vec{N} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$ @ closest

$$\vec{N} \parallel \vec{r} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$x_0 = a \cdot 4$

$y_0 = a \cdot 3$

$z_0 = a\,5$

$7 = 4x_0 + 3y_0 + 5z_0 = a(16+9+25) = 50a$

$a = \frac{7}{50}$

Closest and farthest points

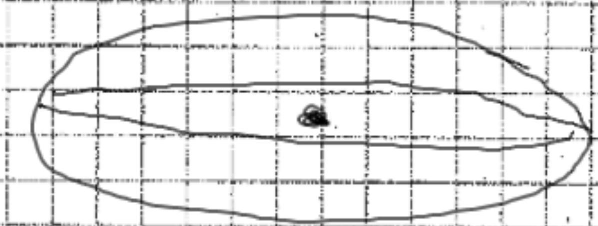
1. How about a plane that misses the origin?
2. How about an ellipse or some such curved surface?

Closest and farthest points

1. How about a plane that misses the origin?
2. How about an ellipse or some such curved surface?

$$1 = \frac{x^2}{4} + y^2 + z^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \parallel \begin{bmatrix} x/2 \\ 2y \\ 2z \end{bmatrix}$$



$$x_0 = a\,(x_0/2) \qquad z_0 = a\,2z_0$$

$$y_0 = a\;2y_0$$

$$1 = a/2$$

$$1 = 2a$$

Closest and farthest points

1. How about a plane that misses the origin?

$$\sin(x)\sin(y)$$

$$= \left(x - \frac{x^3}{3!} + \cancel{O(x^5)}\right)\left(y - \frac{y^3}{3!} + O(y^5)\right)$$

$$\frac{x^5}{5!} + O(x^7))($$

Examples - Find and classify critical points

- $x^3 + (x - y)^2$

Examples - Find and classify critical points

- $x^3 + (x - y)^2$

1 $\nabla f = \begin{bmatrix} 3x^2 + 2(x-y) \\ -2(y-x) \end{bmatrix}$

cp's satisfy

$0 = 3x^2 + 2x - 2y$

$= 2(y-x)$

$x = y$ &

$3x^2 = 0$

$x = 0$

2 $\mathcal{H}$ $\begin{bmatrix} 6x+2 & -2 \\ -2 & 2 \end{bmatrix}$

@ cp

$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\mathcal{D} = 0$

Examples - Find and classify critical points

- $4x^3 + y^2 - 12x^2 - 36x$
  - (3, 0) is a local minimum (Hessian matrix has positives on diagonal and positive determinant.)
  - (-1,0) is a saddle. (Hessian matrix has negative determinant.)

Examples

- $f(x,y,z) = x^2 + 4y^2 - 8y + 3z^2 + xy - yz + 2$
- $f(x,y,z) = x^2 - 4y^2 - 8y + 3z^2 + xy - yz + 2$
- $f(x,y,z) = 7 - x^2 - 4y^2 - 8y - 3z^2 + xy - yz$

Quick and dirty Taylor

$$f(x,y,z) = \frac{1+xy}{1-\boxed{xz}}$$

$$\frac{1}{1-t} = 1+t+t^2+\dots$$

$$= (1+xy)(1+xz+x^2z^2+x^3z^3+\dots)$$

$$= 1+xy+xz+x^2z^2+x^2yz + \text{deg } 3 \text{ or hots}$$

Quick and dirty Taylor

$$\sin(x)\sin(y)$$

$$= \left(x - \frac{x^3}{3!} + O(x^5)\right)\left(y - \frac{y^3}{3!} + O(y^5)\right)$$

$$\frac{x^5}{5!} + O(x^7)\Big)\Big($$

Example

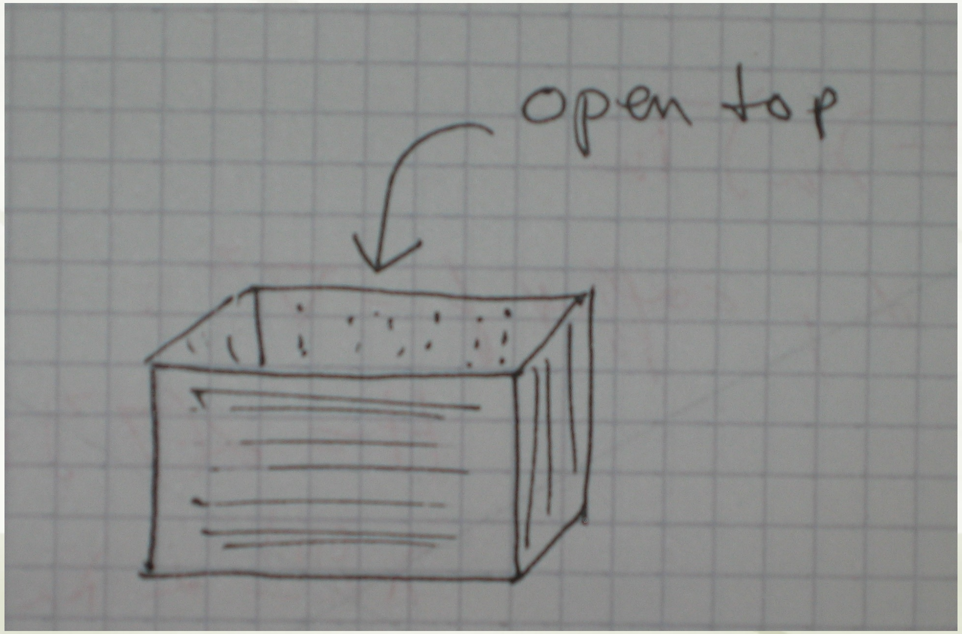
- How can you minimize the amount of wood needed to make a rectangular box, with no top, holding 1 cubic meter?

Absolute max and min

- Here is the algorithm for a closed, bounded region and continuous function:
  - Calculate gradient
  - List all critical points
    - Optional: Use Hessian test to eliminate some candidates.
  - Also check the boundary points
    - (more on this subject in future lecture)

A word problem – "modeling"

- How can you minimize the amount of wood needed to make a rectangular box, with no top, holding 1 cubic meter?



A word problem – "modeling"

- How can you minimize the amount of wood needed to make a rectangular box, with no top, holding 1 cubic meter?
- The *objective function* is $xy+2\,xz + 2\,yz$.
- The *constraints* are: $0 \leq x, y, z$, and $xyz=1$.
- Use one constraint to eliminate one variable.

One option: Use the constraint to reduce the dimension. Substitute $z = 1/xy$ to get an objective function of the form

$$f(x,y) = x\,y + 2/x + 2/y\ .$$

Although this region is unbounded, we intuitively know that the box will have finite dimensions.

As for the constraints $0 \leq x,y,z$, we also know that on the boundary: $\{x=0\}$ or $\{y=0\}$ or $\{z=0\}$, the box would have 0 volume.

The good values (x,y) are in the interior, and therefore are a critical point.

One option: Use the constraint to reduce the dimension. Substitute $z = 1/xy$ to get an objective function of the form

$$f(x,y) = x\,y + 2/x + 2/y\ .$$

The gradient is

$$\nabla f(x,y) = (y - 2/x^2)\mathbf{i} + (x - 2/y^2)\mathbf{j}$$

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- Solving gives $x = y = 2^{1/3}$ as the only critical point.

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla g$ on U, then

  $$f(\mathbf{x}) = g(\mathbf{x}) + C.$$

- How about an example?
- $\mathrm{Arctan}(y/x)$ vs. $\mathrm{Arccos}(x/(x^2+y^2)^{1/2})$

```
In[1]:= {D[ArcTan[y/x], x], D[ArcTan[y/x], y]}
```

$$\text{Out[1]=}\ \left\{-\frac{y}{x^2\left(1+\frac{y^2}{x^2}\right)},\ \frac{1}{x\left(1+\frac{y^2}{x^2}\right)}\right\}$$

```
In[2]:= {D[ArcCos[x/Sqrt[x^2 + y^2]], x],
         D[ArcCos[x/Sqrt[x^2 + y^2]], y]}
```

$$\text{Out[2]=}\ \left\{-\frac{-\frac{x^2}{(x^2+y^2)^{3/2}}+\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{1-\frac{x^2}{x^2+y^2}}},\ \frac{x\,y}{(x^2+y^2)^{3/2}\sqrt{1-\frac{x^2}{x^2+y^2}}}\right\}$$

```
In[3]:= Simplify[%% - %]
```

$$\text{Out[3]=}\ \left\{-\frac{y}{x^2+y^2}+\frac{\sqrt{\frac{y^2}{x^2+y^2}}}{\sqrt{x^2+y^2}},\ x\left(-\frac{y}{\sqrt{\frac{y^2}{x^2+y^2}}\ (x^2+y^2)^{3/2}}+\frac{1}{x^2+y^2}\right)\right\}$$