MATH 2411 - Harrell

The Three ∇s

Lecture 14

Grad, Curl, and Div

But first, a contest…

Math stories

- Two mathematicians meet in the Skiles Building. The first asks the second how her family is, and the second answers: "They're great. My three sons all had birthdays last week. The sum of their ages is 13. The product of their ages is the same as your street number.

- “Hmm...” says the first mathematician. I need to know something more before I can tell you how old your children are." The second answers: "Oh, my eldest son plays the violin." The first mathematician says: “Aha, now I know.”

Vector fields

1. $\mathbf{F}(\mathbf{r}) = x\,\mathbf{i} + y\,\mathbf{j}$. $\mathbf{F}(\mathbf{r}) = \nabla(x^2+y^2)/2$ at each point

2. $\mathbf{F}(x) = y\,\mathbf{i} + x\,\mathbf{j}$. $\mathbf{F}(\mathbf{r}) = \nabla\, xy$

3. $\mathbf{F}(x) = -\,y\,\mathbf{i} + x\,\mathbf{j}$. $\mathbf{F}(\mathbf{r})$ is not a gradient.

Favorite functions of the day

- Scalar function
  - xy, $|\mathbf{r}|^3$
- Vector fields
  - $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, a.k.a. $\mathbf{r}$.
  - $-y\,\mathbf{i} + x\,\mathbf{j} + 0\,\mathbf{k}$.
  - $-\mathbf{r}/|\mathbf{r}|^3$. *Why is this one important?*

From Wolfram MathWorld

Discrete Mathematics
Foundations of Mathematics
Geometry
History and Terminology
Number Theory
Probability and Statistics
Recreational Mathematics
Topology

Alphabetical Index
Interactive Entries
Random Entry
New in *MathWorld*

MathWorld Classroom

About *MathWorld*
Send a Message to the Team

Order book from Amazon

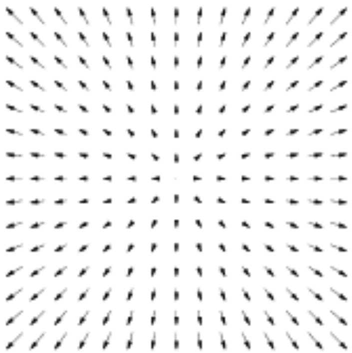
Last updated:
12,720 entries
Tue Oct 23 2007

Created, developed, and nurtured by Eric Weisstein at Wolfram Research

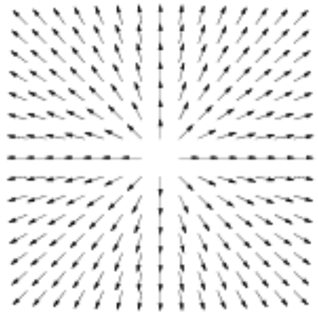
Vector Field

COMMENT On this Page   EXPLORE THIS TOPIC IN The *MathWorld* Classroom   DOWNLOAD *Mathematica* Notebook

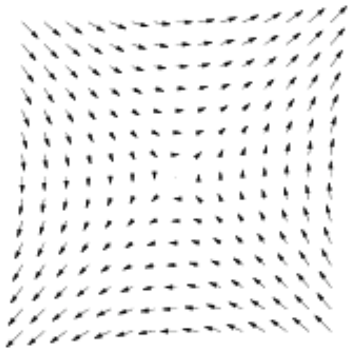
$\{x, y\}$



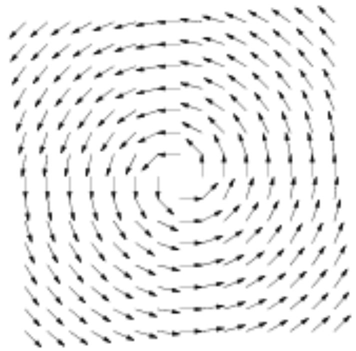
$\left\{\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right\}$



$\{y, x\}$



$\{-y, x\}$



A vector field is a map $\mathbf{f}: \mathbb{R}^n \mapsto \mathbb{R}^n$ that assigns each $\mathbf{x}$ a vector function $\mathbf{f}(\mathbf{x})$. In French, a vector field is called "un champ." Several vector fields are illustrated above. A vector field is uniquely specified by giving its divergence and curl within a region and its normal component over the boundary, a result known as Helmholtz's theorem (Arfken 1985, p. 79).

Integrator
Tones
Functions Site
Wolfram Science
more...

Learn more about vector fields in *Modern Differential Geometry of Curves and Surfaces with Mathematica*.

- Scalar function
  - xy, $|\mathbf{r}|^3$
- Vector fields
  - $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, a.k.a. $\mathbf{r}$.
  - $-y\,\mathbf{i} + x\,\mathbf{j} + 0\,\mathbf{k}$.
  - $-\,\mathbf{r}/|\mathbf{r}|^3$.

If a v.f. (esp. a force)

is a gradient of a scalar

fn, it is said to

be conservative.

Vector fields

1. $\mathbf{F}(\mathbf{r}) = x\,\mathbf{i} + y\,\mathbf{j}$. $\mathbf{F}(\mathbf{r}) = \nabla(x^2+y^2)/2$ at each point

2. $\mathbf{F}(x) = y\,\mathbf{i} + x\,\mathbf{j}$. $\mathbf{F}(\mathbf{r}) = \nabla\, xy$

3. $\mathbf{F}(x) = -y\,\mathbf{i} + x\,\mathbf{j}$. $\mathbf{F}(\mathbf{r})$ is not a gradient.

3 or more vars

$P(x,y,z)\,\hat{i} + Q(\ldots)\,\hat{j} + R(x,y,z)\,\hat{k}$

$P = \dfrac{\partial f}{\partial x}$

$P_y = Q_x$

$P_z = R_x$

$Q_z = R_y$

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$$

U is the “potential energy.” **F** is a “conservative force.”

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$$

If **r**(t) is a curve, then E (t) is a function of t.

$$E = \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r})$$

In principle.

Total energy = kinetic + potential

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \qquad E = \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r})$$

$$\frac{dE}{dt} = \frac{1}{2} m 2 \vec{v} \cdot \dot{\vec{v}} + \nabla U(\vec{r}) \cdot \dot{\vec{r}}$$

$$= \vec{v} \cdot (m\dot{\vec{v}} + \nabla U)$$

$$\cdot = \frac{d}{dt} \cdots$$

$$= \vec{v} \cdot (m\vec{a} - \vec{F})$$

$$= 0$$

$$\lim \left(\cancel{U(r(t))} - U(r(t)) + \nabla U \cdot \left(\frac{r(t+h) - r(t)}{h} \right) \right.$$

hots

Conservation of energy

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \qquad E = \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r})$$

$$\frac{d}{dt} U(\vec{r}(t))$$

$$= \lim_{h \to 0} \frac{U(\vec{r}(t+h)) - U(\vec{r}(t))}{h}$$

$$U(r(t+h)) = U(\vec{r} + \Delta\vec{r})$$

$$\simeq U(\vec{r}) + \boxed{\nabla U} \cdot \Delta\vec{r}$$

Application: Escape velocity

How fast do you need to blast off to be lost in space? *" Ground control to Major Tom...."*

$r_e \approx 6.3 \times 10^6 \, m$

Grav. P.E. $-\dfrac{GmM}{r} =: U$

$$E = \frac{1}{2} m|v|^2 + U$$

$$v_e = \sqrt{\frac{2GM}{r_e}}$$

(r → large)

$$\frac{1}{2}\cancel{m} v_e^2 = \boxed{\frac{G\cancel{m}M}{r_e^2}} \, r_e$$

Differentiation and vector fields

Maxwell's equations

Wikipedia article on *Maxwell's Equations*:

General case

[edit]

The Equations are given in SI units. See below for CGS units.

Name	Differential form	Integral form
Gauss's law:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_S}{\epsilon_0}$
Gauss' law for magnetism (absence of magnetic monopoles):	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{d\Phi_{B,S}}{dt}$
Ampère's Circuital Law (with Maxwell's correction):	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \epsilon_0 \frac{d\Phi_{E,S}}{dt}$

The following table provides the meaning of each symbol and the SI unit of measure:

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\nabla \cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla \times$	the curl operator	

Grad, Curl, and Div

- Just for fun, think of ∇ as a vector "operator" with components
  - $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.
- And do with it what you like to do with vectors.

New derivatives

- Derivative in out notation
  - ∇ = grad — gradient
  - ∇• = div — divergence
  - ∇× = curl — curl a.k.a. rot

New derivatives

Derivative	in	out	notation
∇f = grad f	scalar		gradient
$\nabla \bullet \mathbf{v}$ = div **v**	vector		divergence
$\nabla \times \mathbf{v}$ = curl **v**	vector		curl
			a.k.a. rot **v**

New derivatives

Derivative	in	out	notation
∇f = grad f	scalar	vector	gradient
$\nabla \bullet \mathbf{v}$ = div **v**	vector	scalar	divergence
$\nabla \times \mathbf{v}$ = curl **v**	vector	vector	curl a.k.a. rot **v**

Grad, curl, and div

- grad. ∇f
  - The direction uphill and the slope
  - Critical points
  - Normal vectors and tangent planes

Grad, curl, and div

- div. $\nabla \bullet \mathbf{v}$

  - Quantifies the tendency of a vector field to spread.

  - Related to flux (stay tuned)

Grad, curl, and div

- curl. $\nabla \times \mathbf{v}$
  - Quantifies the tendency of a vector field to swirl.
  - Related to flux (stay tuned)

New rules

- Linear rules
- Product rules
- Chain rules
- Higher derivatives
  - Laplacian $\nabla^2 = \nabla \bullet \nabla$
  - $\nabla \times \nabla f =$
  - $\nabla \bullet \nabla \times \mathbf{v} =$

Favorite functions of the day

- Scalar function
  - xy, $|\mathbf{r}|^3$
- Vector fields
  - $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, a.k.a. $\mathbf{r}$.
  - $-y\,\mathbf{i} + x\,\mathbf{j} + 0\,\mathbf{k}$.
  - $-\mathbf{r}/|\mathbf{r}|^3$. *Why is this one important?*

$$u(\vec{r}) = (u_1(\vec{r}), u_2(\vec{r}), u_3(\vec{r}))$$

$$u \cdot \nabla f = \left(u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}\right) f$$

$$= u_1 f_x + u_2 f_y + u_3 f_z$$