MATH 2411 - Harrell

Slicing, dicing, and integration

Lecture 15

This week's learning plan

- A closer look at grad, curl, and div.
  (even in fancy variables)
- Integration – or is it disintegration?
  - What's it all about in 3D
  - How do you calculate integrals?

Other matters

- HW due day?

A Monday

B Tuesday √

C Wednesday

D Thursday

Grad, Curl, and Div

Congratulations to Riesling M for her solution of last week's puzzle

while she probably wasn't paying rapt attention to my lecture

- Two mathematicians meet in the Skiles Building. The first asks the second how her family is, and the second answers: "They're great. My three sons all had birthdays last week. The sum of their ages is 13. The product of their ages is the same as your street number.

- "Hmm..." says the first mathematician. I need to know something more before I can tell you how old your children are." The second answers: "Oh, my eldest son plays the violin." The first mathematician says: "Aha, now I know."

Clicker Quiz

Consider a scalar function like $f(x,y,z) := xyz$ and a vector field like $\mathbf{v}(x,y,z) := xz\,\mathbf{i} - yz\,\mathbf{k}$. Which of the following is true?

A $\nabla \bullet f$ is nonsense and $\nabla \times \mathbf{v}$ is a scalar field.

B $\nabla \times \mathbf{v}$ is nonsense and $\nabla \bullet f$ is a scalar field.

C ∇f is nonsense and $\nabla \times \mathbf{v}$ is a vector field.

D $\mathbf{f} \bullet \nabla$ is nonsense and $\mathbf{v} \times \nabla f$ is a vector field. √

New rules

- Linear rules
- Product rules
- Chain rules
- Higher derivatives
  - Laplacian $\nabla^2 = \nabla \bullet \nabla$
  - $\nabla \times \nabla f =$
  - $\nabla \bullet \nabla \times \mathbf{v} =$

Details about the rules are to be found in the text. Remember: 1. The basic rules look as much as possible like those of KG calculus. 2. Keep what's left on the left and what's right on the right. 3. For the chair rule, the most reliable thing is to look at each input and each output component, and sum up KG-looking expressions for all the ways of getting from the first input to the last output.

Grad, curl, and div in interesting coordinates

- We can change variables in and expression like $\nabla = \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z$. Cylindrical isn't too hard:
  - $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, and

    $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$,

    so

    $\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$, and

    $\mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta$, and

Grad, curl, and div in interesting coordinates

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  - $\partial_x = (\partial\theta/\partial x)\,\partial_\theta + (\partial r/\partial x)\,\partial_r$

    $= -((\sin\theta)/r)\,\partial_\theta + (\cos\theta)\partial_r$, and

    $\partial_y = (\cos\theta)/r\,\partial_\theta + (\sin\theta)\partial_r$

    Note: This takes a calculation, and at the end you need to remember to rewrite everything in terms of the new variables r,θ.

Grad, curl, and div in interesting coordinates

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  - $\nabla = \mathbf{e}_r\,\partial_r + (\mathbf{e}_\theta/r)\,\partial_\theta + \mathbf{e}_z\,\partial_z$ (where $\mathbf{e}_z = \mathbf{k}$)

Gradient, Divergence, and Curl in Cylindrical and Spherical Coordinates

By similar arguments to Example 4, we find that the curl in cylindrical coordinates is given by

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.$$

We can find other important vector quantities expressed in different coordinate systems. For example, the chain rule shows that the gradient in cylindrical coordinates is

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z,$$

and in Section 8.4 we will establish related techniques that give the following formula for the divergence in cylindrical coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z}(rF_z)\right].$$

Corresponding formulas for gradient, divergence, and curl in spherical coordinates are

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\phi) + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

and

$$\nabla \times \mathbf{F} = \left[\frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\theta) - \frac{1}{\rho\sin\phi}\frac{\partial F_\phi}{\partial \theta}\right]\mathbf{e}_\rho + \left[\frac{1}{\rho\sin\phi}\frac{\partial F_\rho}{\partial \theta} - \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\theta)\right]\mathbf{e}_\phi + \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\phi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right]\mathbf{e}_\theta,$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_\theta$ are as shown in Figure 8.2.10 and where $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_\theta\mathbf{e}_\theta$.

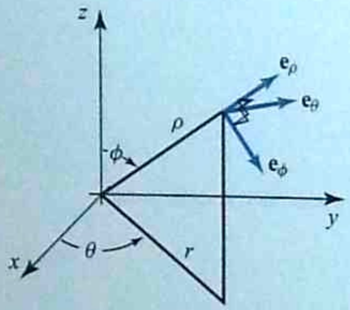

figure **8.2.10** Orthonormal vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$ associated with spherical coordinates.

This one isn't what you might have predicted. Why?

Gradient, Divergence, and Curl in Cylindrical and Spherical Coordinates

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$$\nabla \times \mathbf{F} = \left[\frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\theta) - \frac{1}{\rho\sin\phi}\frac{\partial F_\phi}{\partial \theta}\right]\mathbf{e}_\rho + \left[\frac{1}{\rho\sin\phi}\frac{\partial F_\rho}{\partial \theta} - \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\theta)\right]\mathbf{e}_\phi + \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\phi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right]\mathbf{e}_\theta,$$

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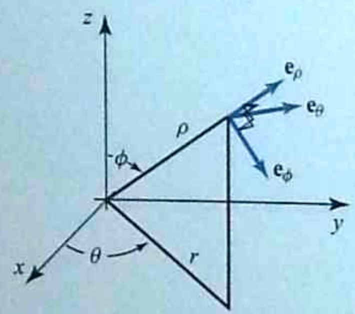


figure 8.2.10 Orthonormal vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$ associated with spherical coordinates.

The Laplacian and some of the great PDEs

Laplacian

- Laplace's equation for "harmonic fns":
  - $\Delta u = \nabla^2 u = 0$.
    - Equilibrium membrane, electric potential
- Heat or diffusion equation
  - $u_t = k \nabla^2 u$.
    - Temperature, density of dye
- Wave equation
  - $u_{tt} = c^2 \nabla^2 u$.
    - Sound, light

A *sharp!* break to a new subject:

Integration

and the culinary arts?

Puzzle break!

- $0 = 0 \quad + \quad 0 \quad + \quad 0 \quad + \ \ldots\ldots$

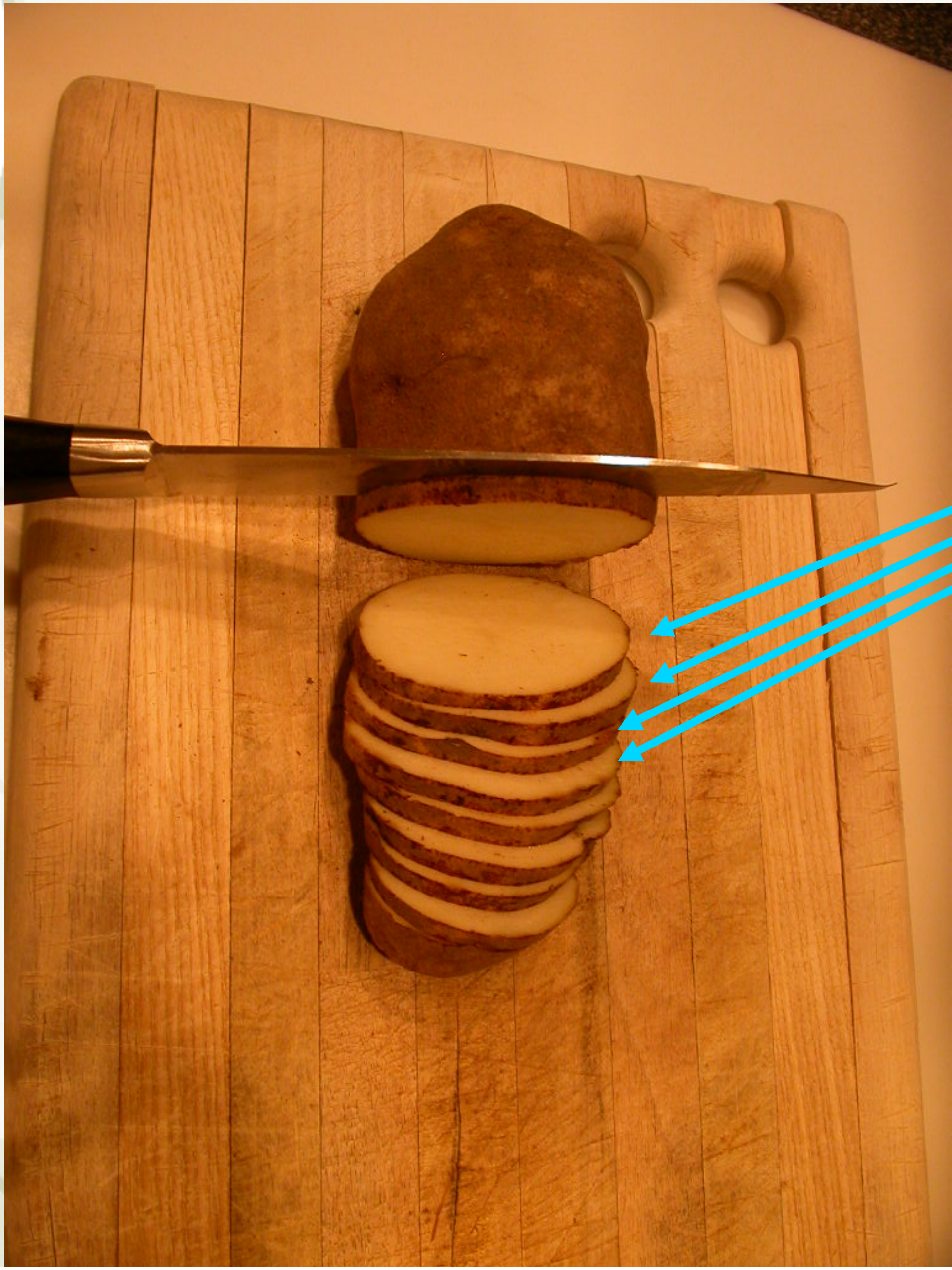
  $= (1 - 1) + (1 - 1) + (1 - 1) + \ \ldots\ldots$

  $= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots..$

  $= 1 + \quad 0 \quad + \quad 0 \quad + \ \ldots$

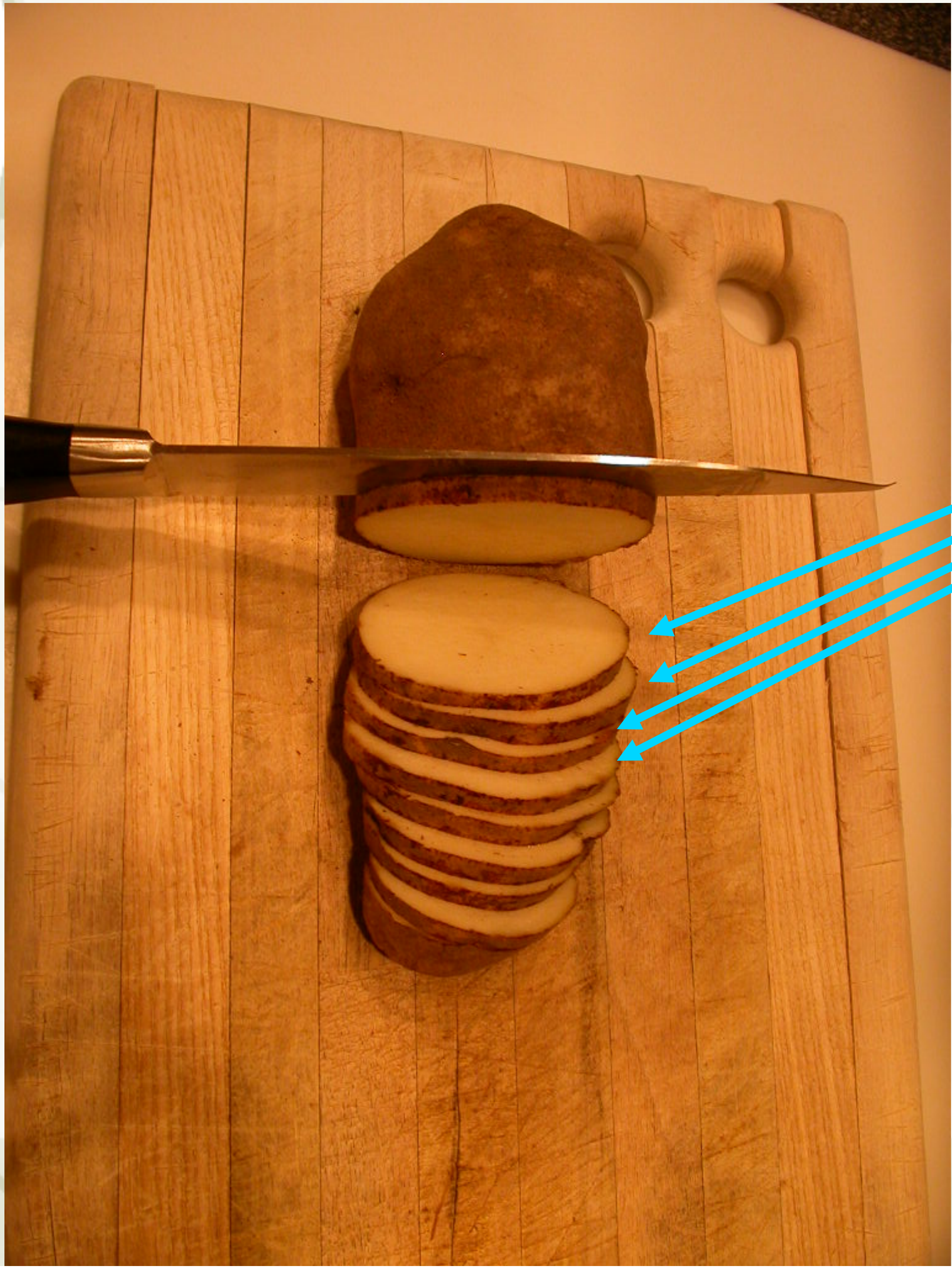
  $= 1$

Visuals for volumes



Each slice has width

Δx

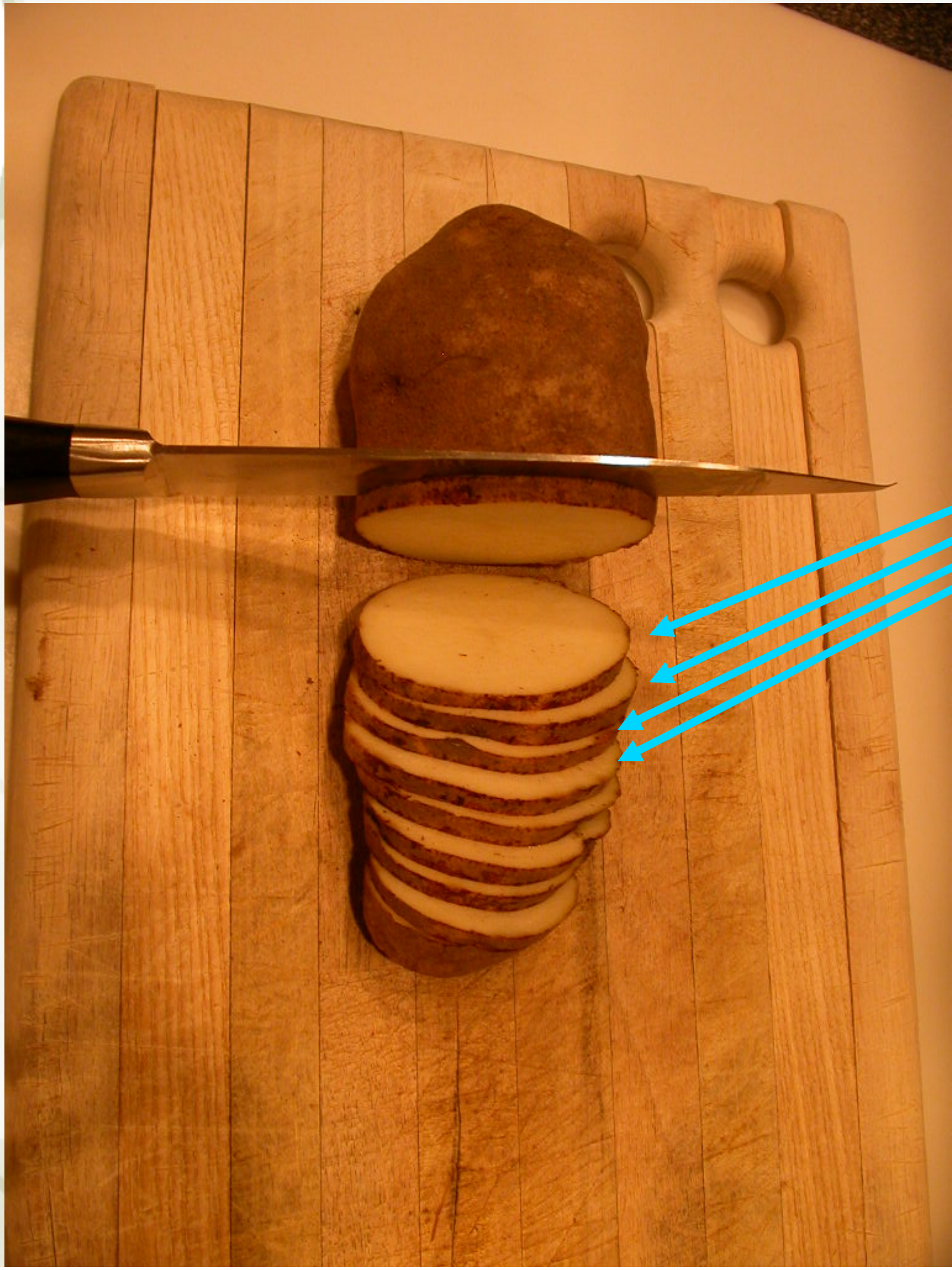


Each slice has width

Δx

Each slice has volume

$A(x)\Delta x$



Each slice has width

Δx

Each slice has volume

$A(x)\Delta x$

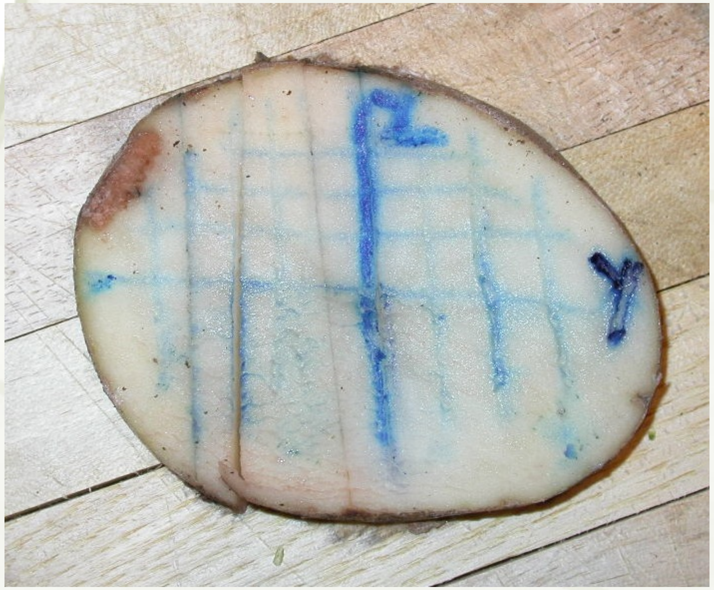
Total volume ≈ a Riemann sum

$\sum_i A(x_i)\Delta x$

Now take a close look at the slices:

Why, there's a little coordinate grid on there! And, and…

Now take a close look at the slices:



So each A(x) is itself an integral of the form ∫z dy.

Suppose the height of a rectangular region is z = h(x,y). The volume is an *iterated integral*,

Integrate over h (x,y) to get A(x).

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This potato has been sliced along the x-axis with width Δx.

What happens when we slice it a second time along the y-axis with width Δy?

FRIES!

The volume of a vertical French fry is (hgt • Δx Δy) (or pretty close)

Double Riemann whammy:

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This is a definite integral plugged into a second definite integral. In other words, it is a delicate kind of limit of a Rieman sum plugged into another limit of a Rieman sum. *Ouch!*

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} h(x_j, y_k) \Delta x \Delta y \right).$$

Anatomy of the notation

$$\int_a^b \left(\int_c^d h(x,y)dy \right) dx = \int_a^b \int_c^d h(x,y)dydx = \int_R h(x,y)dxdy$$

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} a_{jk} \right) = \sum_{j=1}^{N} \sum_{k=1}^{M} a_{jk}$$

Do the inside operation first. (*Does it matter*?)

Double Riemann whammy:

$$Vol(\Omega) = \int_a^b \left(\int_c^d h(x,y)dy \right) dx$$

This is the one we write and calculate.

This is the one we think about and get the computer to calculate.

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{M} h(x_j, y_k)\Delta x \Delta y \right).$$

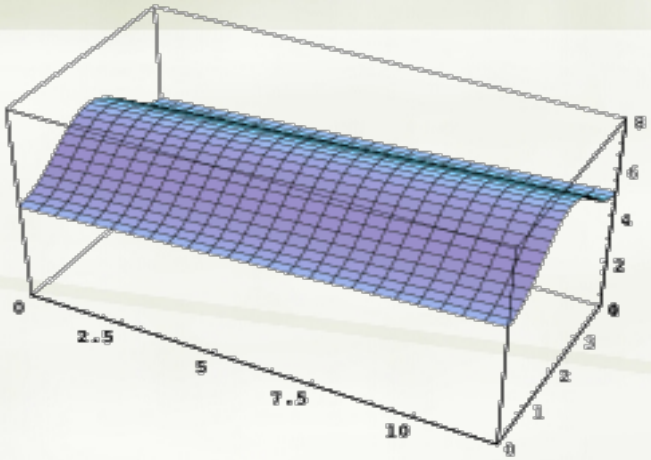
Examples:

A loaf of bread sits in a rectangular pan,

$R = \{0 \le x \le 12, \ 0 \le y \le 4\}$.

The height of the bread at point (x,y) is $5 + (x/12) - \cos(\pi y/2)$ (units are inches). What is the volume of the loaf?

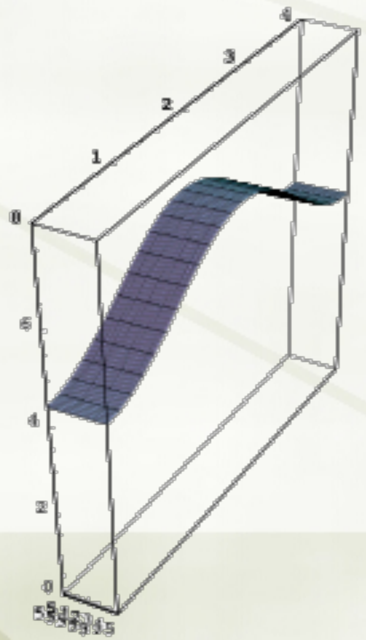
```
In[12]:= Plot3D[5 + (x / 12) - Cos[Pi y / 2], {x, 0, 12},
           {y, 0, 4}, PlotRange → {0, 8}, BoxRatios → {2, 1, 1}]
```




```
Out[12]= - SurfaceGraphics -
```

Here is one slice, of thickness Δx:

```
In[14]:= Plot3D[5 + (x / 12) - Cos[Pi y / 2], {x, 5, 5.5},
           {y, 0, 4}, PlotRange → {0, 8}, BoxRatios → {1, 8, 8}]
```



First integral:

```
In[16]:= Integrate[5 + (x / 12) - Cos[Pi y / 2], {y, 0, 4}]
```

Out[16]= $\frac{60 + x}{3}$

Second integral:

```
In[17]:= Integrate[(60 + x) / 3, {x, 0, 12}]
```

Out[17]= 264

- How about doing it the other way?

First integral:

In[18]:= `Integrate[5 + (x / 12) - Cos[Pi y / 2], {x, 0, 12}]`

Out[18]= $66 - 12\,\mathrm{Cos}\left[\frac{\pi y}{2}\right]$

Second integral:

In[19]:= $\mathtt{Integrate}\left[66 - 12\,\mathrm{Cos}\left[\frac{\pi y}{2}\right],\ \{y,\ 0,\ 4\}\right]$

Out[19]= 264

Practicalities of doing double integrals.

Guido Fubini

From Wikipedia, the free encyclopedia

Guido Fubini (19 January 1879 – 6 June 1943) was an Italian mathematician, known for Fubini's theorem and the Fubini–Study metric.

Born in Venice, he was steered towards mathematics at an early age by his teachers and his father, who was himself a teacher of mathematics. In 1896 he entered the Scuola Normale Superiore di Pisa, where he studied under the notable mathematicians Ulisse Dini and Luigi Bianchi. He gained some early fame when his 1900 doctoral thesis, entitled *Clifford's parallelism in elliptic spaces*, was discussed in a widely-read work on differential geometry published by Bianchi in 1902.

After earning his doctorate, he took up a series of professorships. In 1901 he began teaching at the University of Catania in Sicily; shortly afterwards he moved to the University of Genoa; and in 1908 he moved to the Politecnico in Turin and then the University of Turin, where he would stay for some decades.

During this time his research focused primarily on topics in mathematical analysis, especially differential equations, functional analysis, and complex analysis; but he also studied the calculus of variations, group theory, non-Euclidean geometry, and projective geometry, among other topics. With the outbreak of World War I, he shifted his work towards more applied topics, studying the accuracy of artillery fire; after the war, he continued in an applied direction, applying results from this work to problems in electrical circuits and acoustics.

In 1939, when Fubini at the age of 60 was nearing retirement, Benito Mussolini's Fascists adopted the anti-Jewish policies advocated for several years by Adolf Hitler's Nazis. As a Jew, Fubini feared for the safety of his family, and so accepted an invitation by Princeton University to teach there; he died in New York City four years later.

Guido Fubini

Born	19 January 1879 Venice
Died	6 June 1943 (aged 64) New York
Nationality	Italian
Fields	Mathematics
Alma mater	Scuola Normale Superiore di Pisa
Doctoral advisor	Ulisse Dini Luigi Bianchi
Known for	Fubini's theorem Fubini's theorem on differentiation Fubini–Study metric

Books by G. Fubini [edit]

- Lezioni di analisi matematica (Società Tipografico-Editrice Nazionale, Torino, 1920)

External links [edit]

- O'Connor, John J.; Robertson, Edmund F., "Guido Fubini", *MacTutor History of Mathematics archive*, University of St Andrews.
- ERIC PACE (1997-08-06). "Eugene Fubini, 84; Helped Jam Nazi Radar". *New York Times*. Retrieved 2008-08-10.
- Guido Fubini at the Mathematics Genealogy Project

Practicalities of doing double integrals.

Un integrale doppio
è
*un integrale iterato.**

Practicalities of doing double integrals.

A double integral
is
an iterated integral.

- Let’s take one favorite function, like $f(x,y) = xy$, and integrate it over lots of regions.
  - What does the integral of xy over a region in the first quadrant ($x,y > 0$) represent?
  - What if the region is in the second quadrant ($x < 0$, $y > 0$)?