MATH 2411 - Harrell

# *If you don't like the limits placed upon you, change them!*

Lecture 18

# *In our previous episode…*

- Limits of integration. What is the region of integration in

$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_\_.$$

- How does it look if we integrate in a different order?

$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} \_\_\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_.$$

$$\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_ d\_\_\_\_\_\_.$$

$$\int_0^4 \int_0^{4-x} \int_{\cancel{0}}^{4-x-y} \underline{\hspace{3cm}}\, dz\, dy\, dx.$$

$-1$

$$0 \le y \le \boxed{4-x}$$

$$\int_{-1}^{0} \int_{0}^{4-z} \int_{0}^{4-x} \underline{\hspace{3cm}}\, dy\, dx\, dz.$$

$$+ \int_0^4 \int_0^{4-z} \int_0^{4-x-z}$$

$$\to x \le 4-y-z$$

$$z \le 4-x-y \qquad \boxed{x \le 4-y}$$

$$y \le \boxed{4-x-z}$$

$$y \le \min(4-x,\ 4-x-z)$$

# *Another fun game*

- Limits of integration. What is the region of integration in

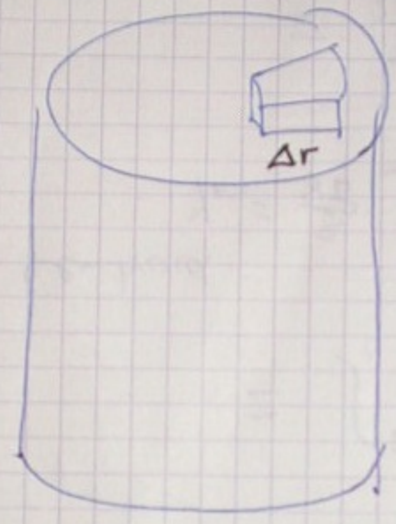
$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \_\_\_\_\_\_\_\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_ d\_\_\_\_\_\_\_.$$

- How does it look if we integrate in a different order?

# *Coordinate systems for grown-ups*

- Cylindrical to Cartesian:
  - $x = r \cos \theta$
  - $y = r \sin \theta$
  - $z = z$
- What's the volume element dV ?

How big is $\Delta V$ ?




in horisontalplane

$$dA = r\,dr\,d\theta$$

height $= dz$

$$dV = r\,dr\,d\theta\,dz$$

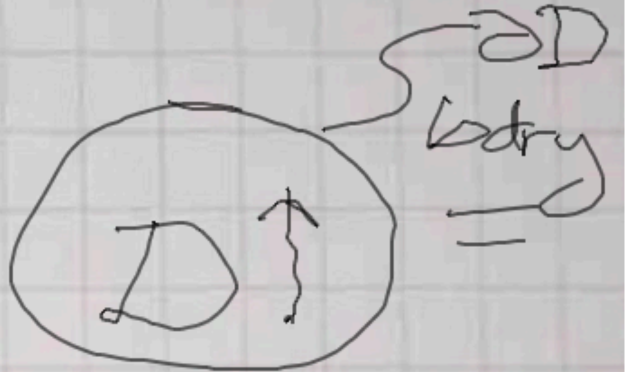
# *Cylindrical examples*

- Volume of sphere
- Volume of 1/4 torus (doughnut)
- Integral of $z\,(x^2 + y^2 + z^2)^{1/2}$, where
  - $x,y > 0$ and
  - $z$ is between $(x^2 + y^2)^{1/2}$ and $(1-x^2-y^2)^{1/2}$
- *What does this region look like?*

- Volume of sphere

## Volume of sphere

$$\int_0^{2\pi}\int_0^{R}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} 1\,dz\,r\,dr\,d\theta$$




$$\partial D = \{ r^2+z^2 = R^2 \}$$

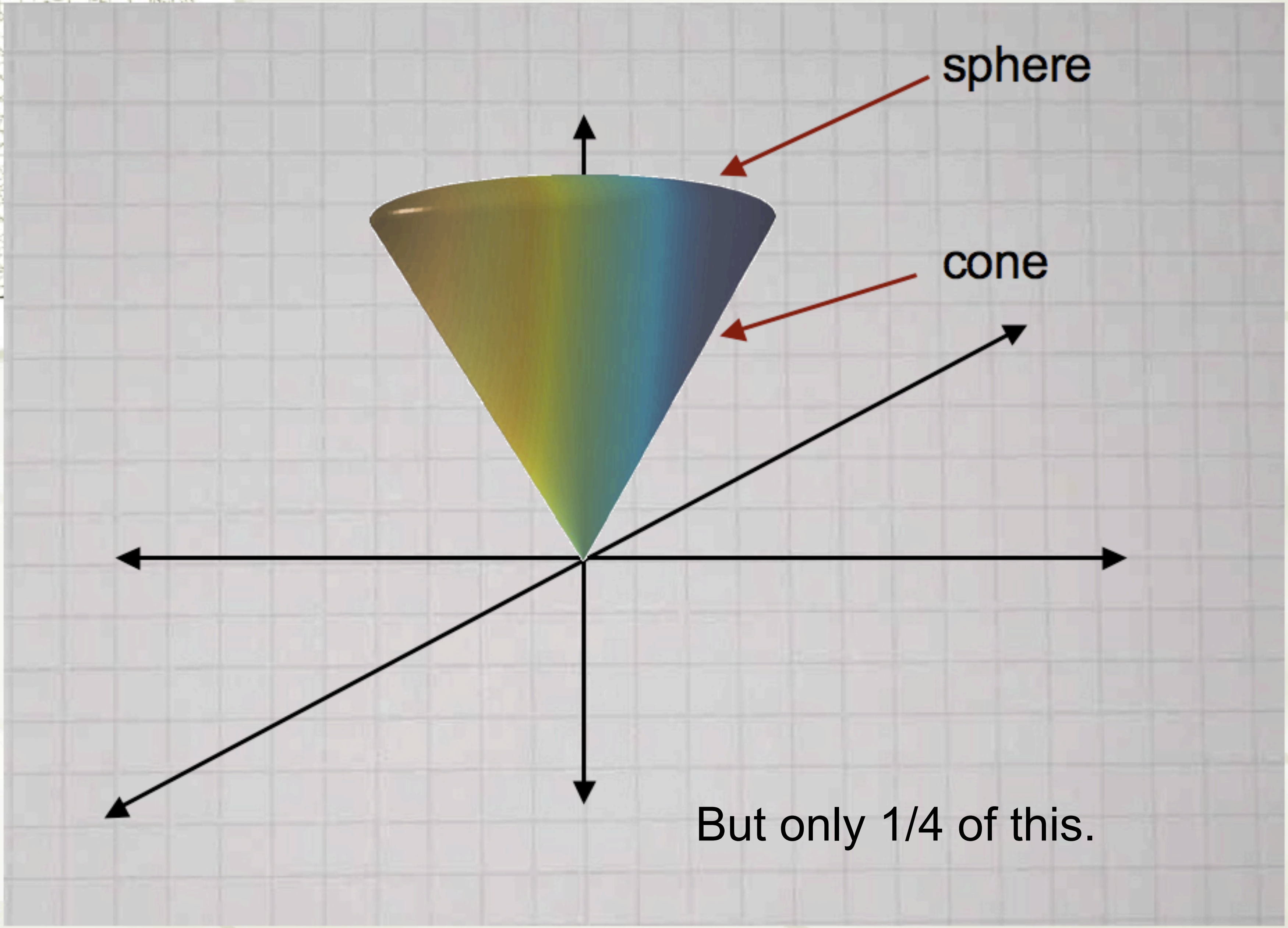
$$2\int_0^{2\pi}\int_0^{R} \sqrt{R^2-r^2}\,r\,dr\,d\theta \qquad u=R^2-r^2$$

$$du=-2r\,dr$$

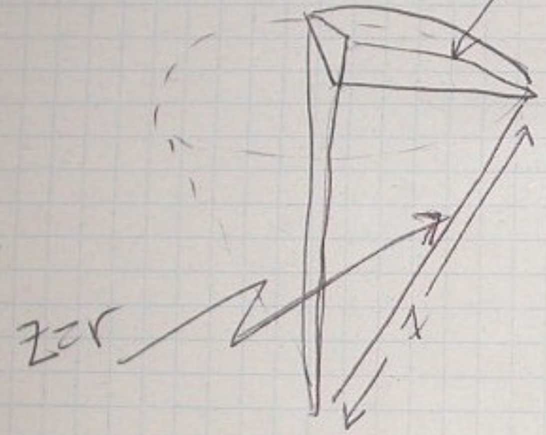
$$= -\int_0^{2\pi}\int_{R^2}^{0} u^{\frac{1}{2}}\,du\,d\theta = \int_0^{2\pi}\left(\frac{2}{3}(R^2)^{3/2}-0\right)$$

$$= \frac{4\pi}{3}R^3$$

- Integral of $z (x^2 + y^2 + z^2)^{1/2}$, where
  - $x,y > 0$ and
  - $z$ is between $(x^2 + y^2)^{1/2}$ and $(1-x^2-y^2)^{1/2}$


sphere
cone
But only 1/4 of this.

# *Cylindrical examples*




Cylindrical volume

$$r \le z \le \sqrt{1-r^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le \frac{1}{\sqrt{2}}$$

$$V = \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} \int_0^{\frac{\pi}{2}} d\theta\, dz\, r\, dr$$

$$= \frac{\pi}{2} \int_0^{\frac{1}{\sqrt{2}}} \left( (1-r^2)^{\frac{1}{2}} r\, dr - r^2 dr \right)$$

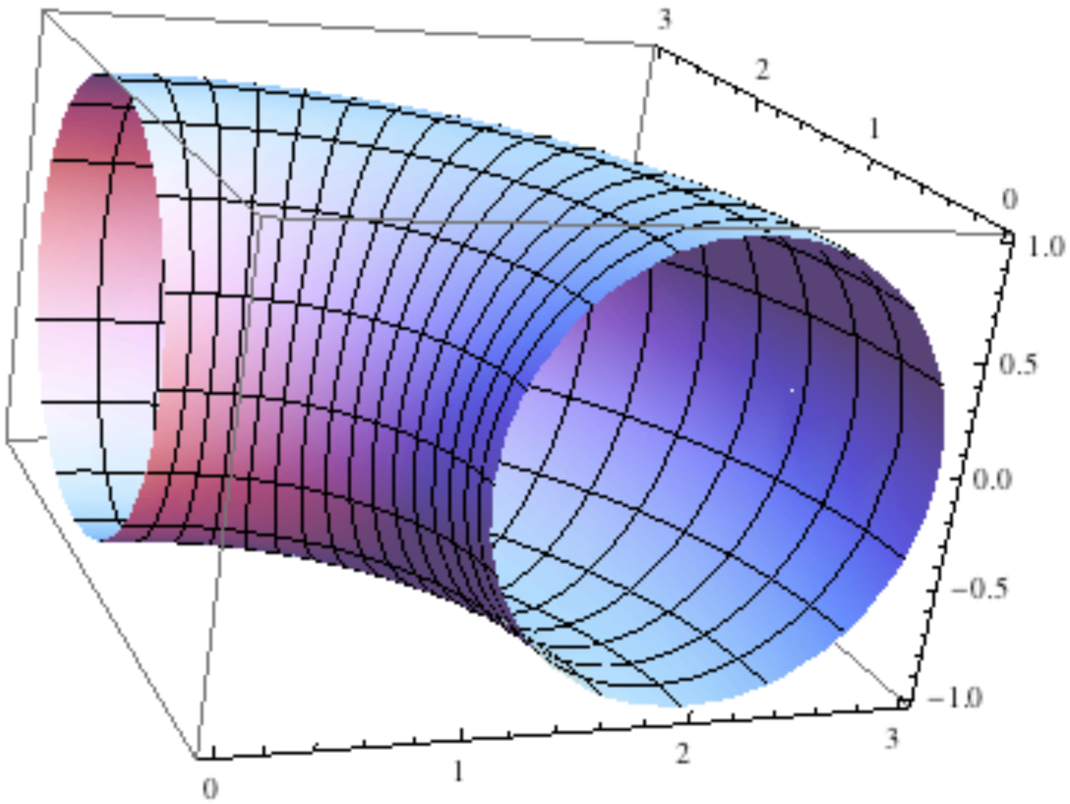
$$V = \frac{\pi}{2}\left(-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3\right)\Bigg|_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{6}\left(1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}}\right) = \boxed{\frac{\pi}{6}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

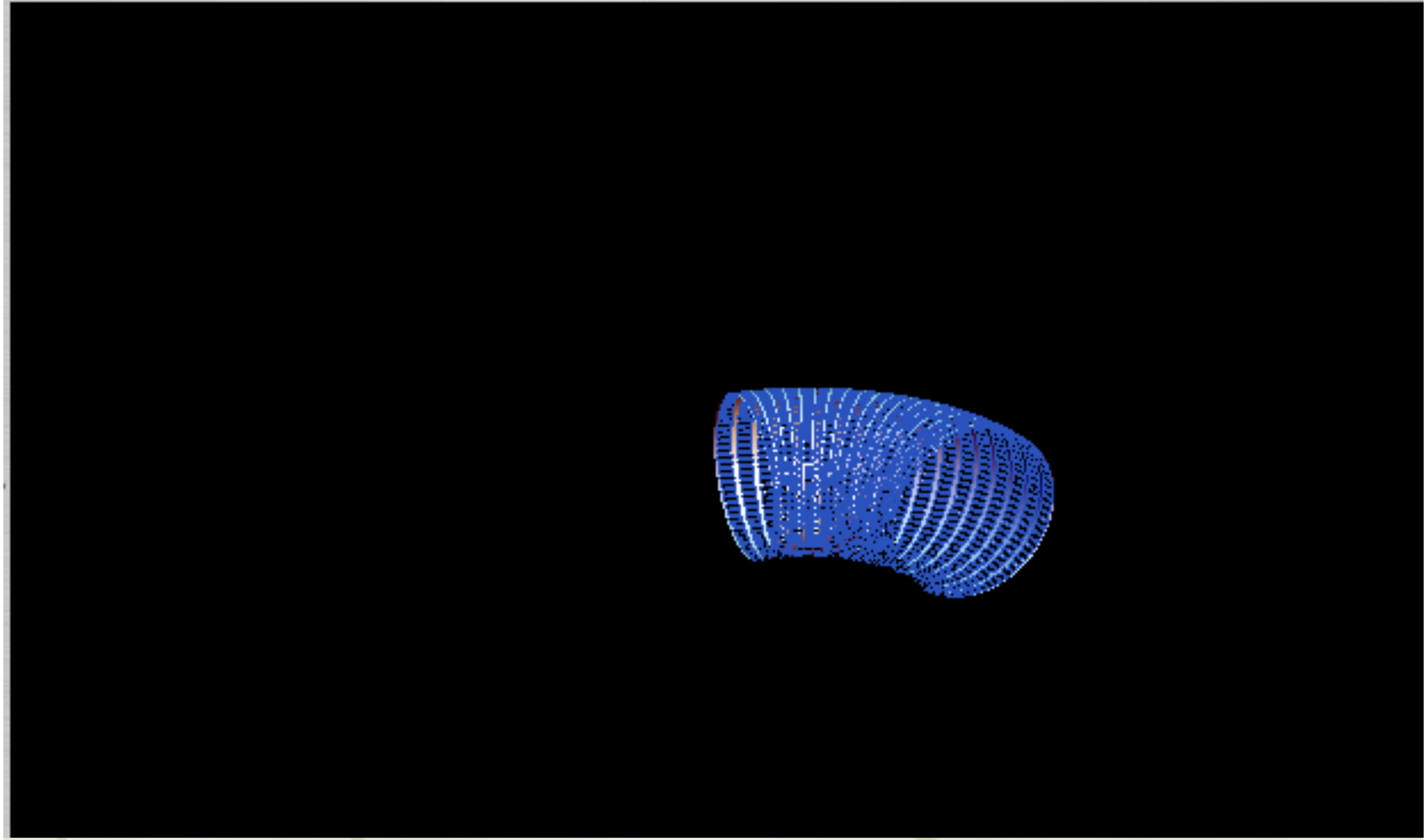
In[3]:= 

```
ParametricPlot3D[{(2 + Cos[t]) Cos[s], (2 + Cos[t]) Sin[s], Sin[t]},
  {s, 0, Pi / 2}, {t, 0, 2 Pi}]
```

Out[3]=



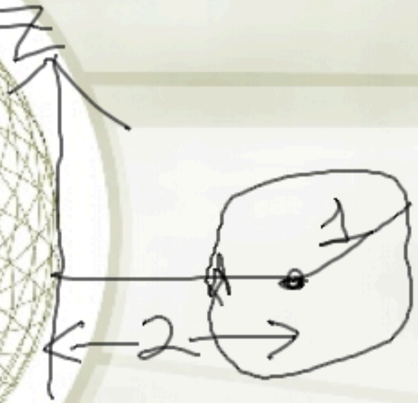
$$\begin{bmatrix} r_0 \\ \phi \\ z \end{bmatrix} = \begin{bmatrix} 2+\cos u \\ t \\ \sin u \end{bmatrix}, t=0\ldots\frac{\pi}{2}, u=0\ldots2\pi$$



# *Cylindrical examples*

- Volume of 1/4 torus (doughnut)

# *Cylindrical examples*




- Volume of 1/4 torus (doughnut)

$$\int_0^{\pi/2}\int_2^4\int_{-\sqrt{1-(r-2)^2}}^{\sqrt{1-(r-2)^2}} 1 \, dz\, r\, dr\, d\theta$$

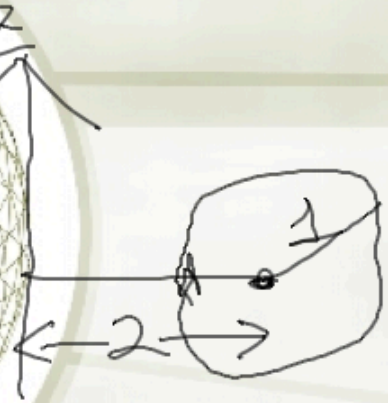
$$= 2\int_0^{\pi/2}\int_1^3 \sqrt{1-(r-2)^2}\, r\, dr\, d\theta$$

$s = r-2$

$r = s+2$

$$= \pi \int_{-1}^{1} \sqrt{1-s^2}\,(s+2)\, ds$$

$$V = 4\pi \int_0^1 \sqrt{1-s^2}\, ds = 4\pi \cdot \frac{\pi}{4} = \pi^2$$




# *Cylindrical examples*

- Volume of 1/4 torus (doughnut)

$$\int_0^{\pi/2} \int_2^4 \int_{-\sqrt{1-(r-2)^2}}^{\sqrt{1-(r-2)^2}} 1 \, dz\, r\, dr\, d\theta$$

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$s = r-2$

$r = s+2$

$$= \pi \int_{-1}^{1} \sqrt{1-s^2}\,(s+2)\, ds$$

# *Estimating integrals you can't actually do.*

If $m \leq f(\mathbf{r}) \leq M$, and $D$ is a region of finite area/volume, hypervolume $V(D)$, then

$$mV(D) \leq \int_D f(\mathbf{r})dV \leq MV(D).$$

# *Estimating integrals you can't actually do.*

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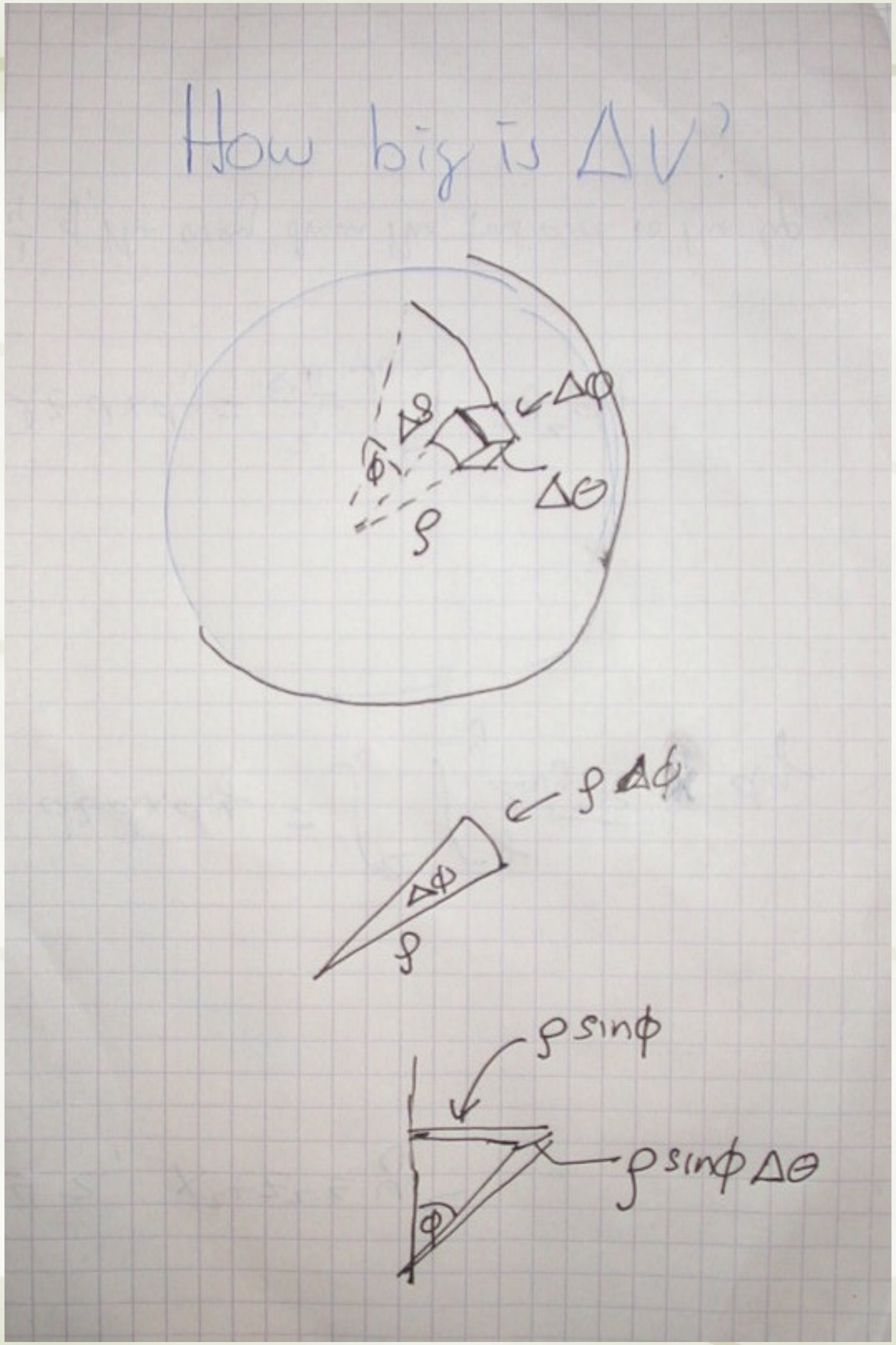
$$\iiint |\sin^{3/2}(xeyz)| \le 1 \cdot \pi^2$$

# *Coordinate systems for grown-ups*

- Spherical = geographic plus $\rho$
  - $\rho$ = distance from origin
  - $\theta$ = polar angle in xy plane = longitude
  - $\phi$ = angle from pole, "colatitude"


How big is ΔV?
Δρ
φ
ρ
Δφ
Δθ
ρΔφ
Δφ
ρ
ρ sinφ
ρ sinφ Δθ
φ

# *Cylindrical examples*

- Volume of ¼ cone with cap
- Volume of sphere
- Volume of a sliced sphere

Spherical volume


$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le \varphi \le \pi/4$$
$$0 \le \rho \le 1$$

no mixed limits!

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 \rho^2 \, d\rho \, \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\left[-\cos\left(\frac{\pi}{4}\right)+1\right] \cdot \frac{\pi}{2} = \frac{\pi}{6}\left(1-\frac{1}{\sqrt{2}}\right)$$

# *Prof. H's special spherical tips*

- Remember, $\theta$ runs from 0 to 2 $\pi$, but $\phi$ runs only from 0 to $\pi$.
- Total "steradians" on the sphere = $4\pi$ = the complete integral of $\sin\phi \, d\phi \, d\theta$
- Very often you want to use the variable $w = \cos\phi$, instead of $\phi$. This variable runs from -1 to 1 and the volume element $dV = \rho^2 \, d\rho \, dw \, d\theta$.

  $dw = - \sin\phi \, d\phi$

# The Great Variable Changer

Carl Gustav Jacob Jacobi
1804-1851

# C. G. J. JACOBI'S

# VORLESUNGEN ÜBER DYNAMIK.

GEHALTEN

AN DER UNIVERSITÄT ZU KÖNIGSBERG IM WINTERSEMESTER 1842—1843
UND NACH EINEM VON C. W. BORCHARDT AUSGEARBEITETEN HEFTE

HERAUSGEGEBEN

VON

A. CLEBSCH.

ZWEITE, REVIDIRTE AUSGABE.

## Sechsundzwanzigste Vorlesung.

### Elliptische Coordinaten.

Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung der richtigen Variablen zu bestehen, zu

From Lecture 26 of the 8-th volume of Jacobi's collected lectures (p. 198): "The greatest difficulty in integrating differential equations seems to consist in introducing the right variables,…"

and not just Cartesian, polar, cylindrical, spherical.

# *Another interesting fact about Jacobi...*

# Mathematics Genealogy Project

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## Carl Gustav Jacob Jacobi

Biography

Ph.D. Humboldt-Universität zu Berlin 1825

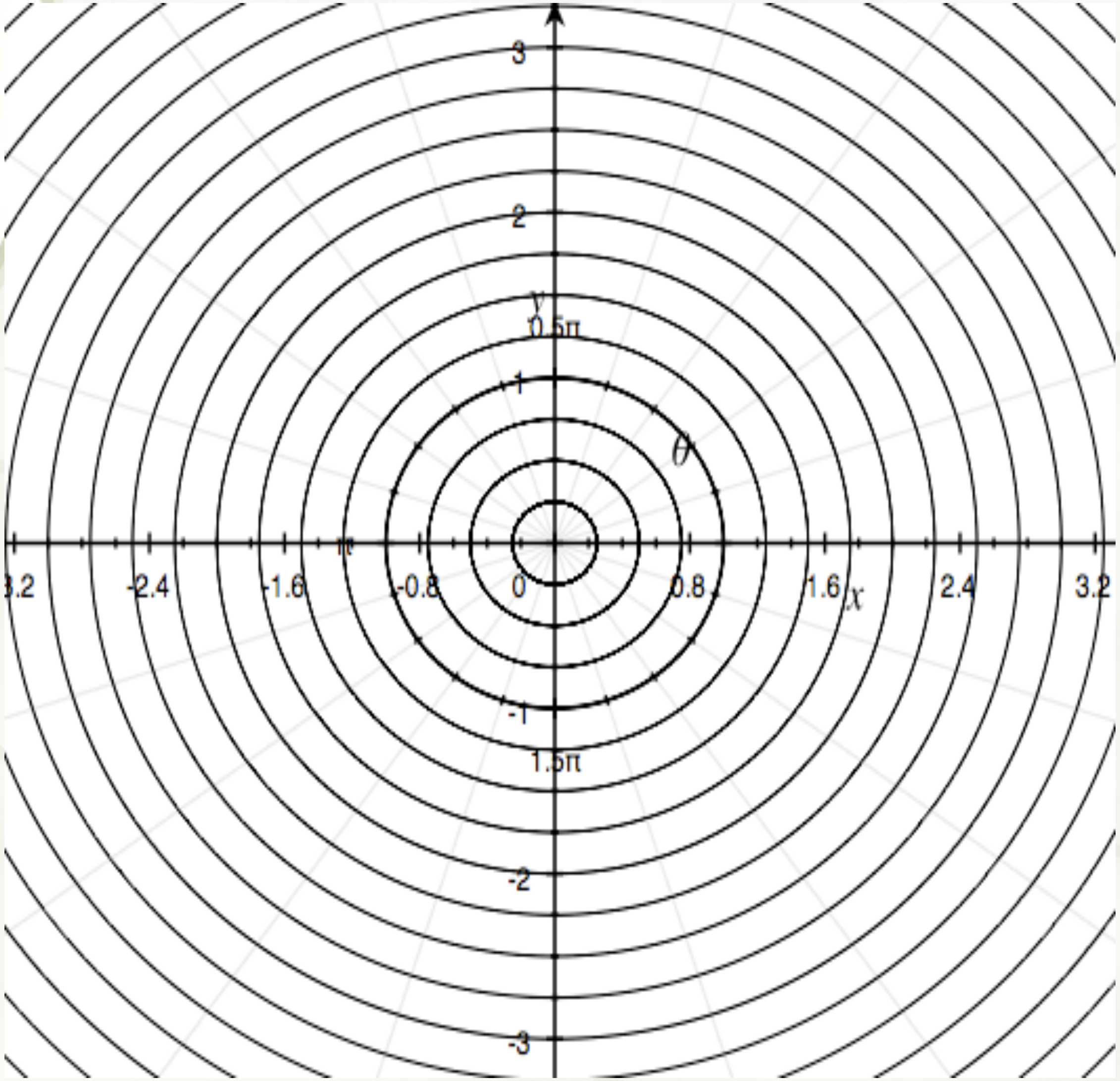
Dissertation: *Disquisitiones Analyticae de Fractionibus Simplicibus*

Advisor: Enno Dirksen

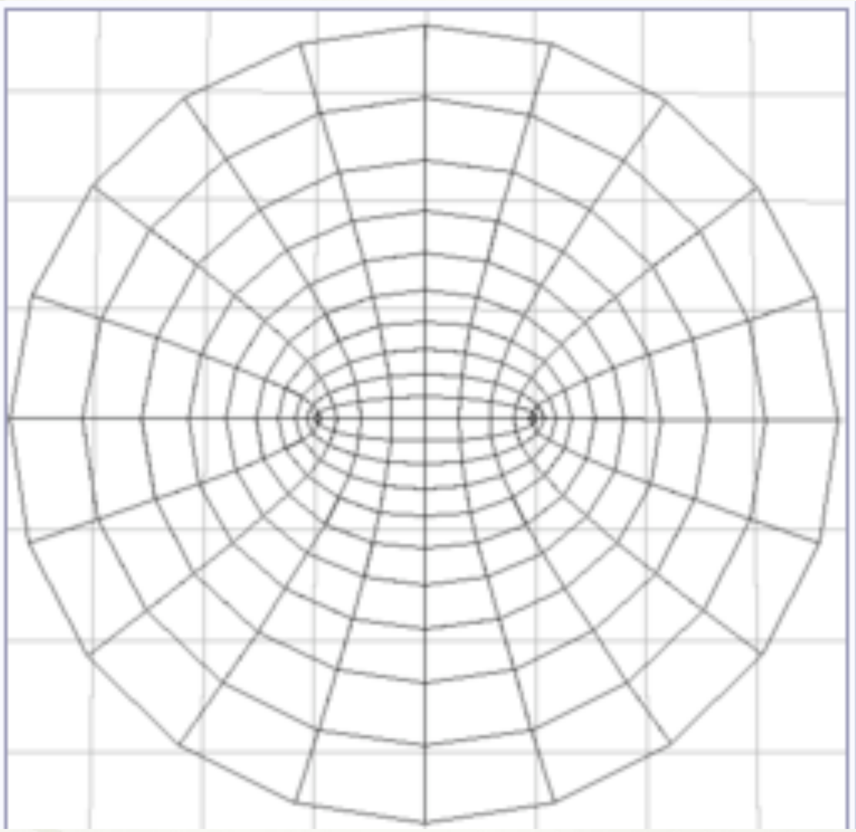
Student(s):
Click here to see the students listed in chronological order.

| Name | School | Year | Descendants |
|---|---|---|---|
| Paul Gordan | Universität Breslau | 1862 | 763 |
| Oswald Hermes | Europa-Universität Viadrina Frankfurt an der Oder | 1849 | |
| Otto Hesse | Universität Königsberg | 1840 | 5626 |
| Friedrich Richelot | Universität Königsberg | 1831 | 5525 |
| Wilhelm Scheibner | Martin-Luther-Universität Halle-Wittenberg | 1848 | 717 |

According to our current on-line database, Carl Jacobi has 5 students and 7420 descendants.




Polar coordinate lines



$x = \cosh u \cos v$

$y = \sinh u \sin v$

But what are u and v in terms of x and y?

Contest!

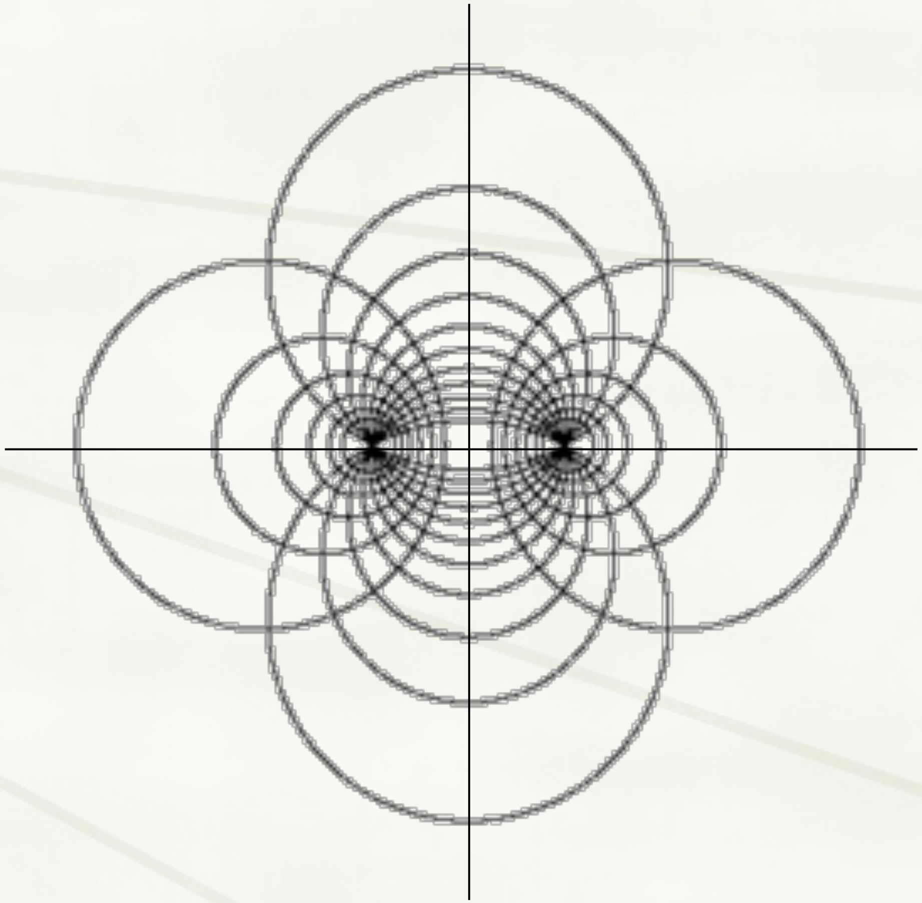
Another example: the bipolar coordinates.

$$x = \sinh(t)/(\cosh(t)-\cos(s))$$

$$y = \sin(s)/(\cosh(t)-\cos(s))$$

*How do you like these coordinate curves?*

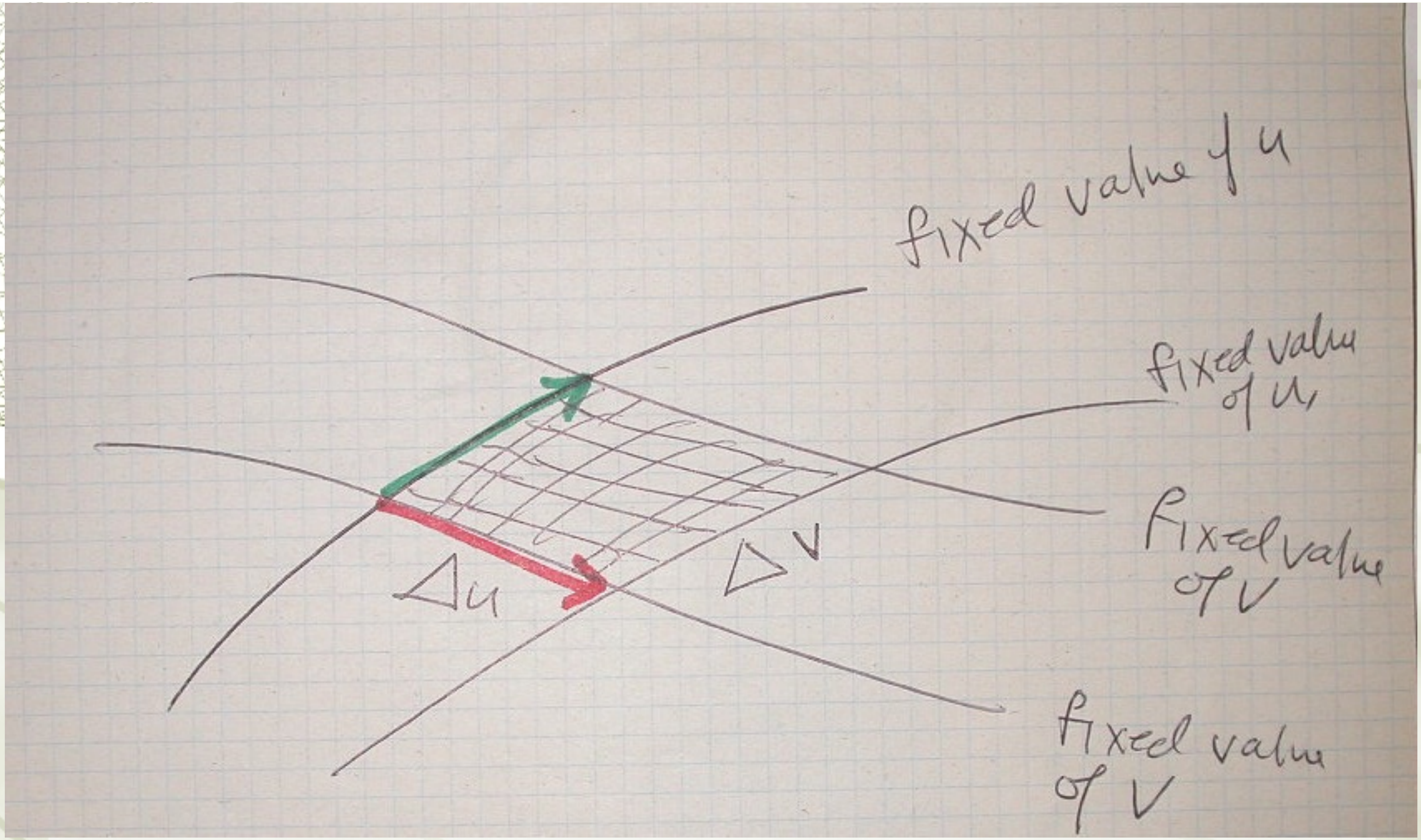
# Bipolar coordinate lines:

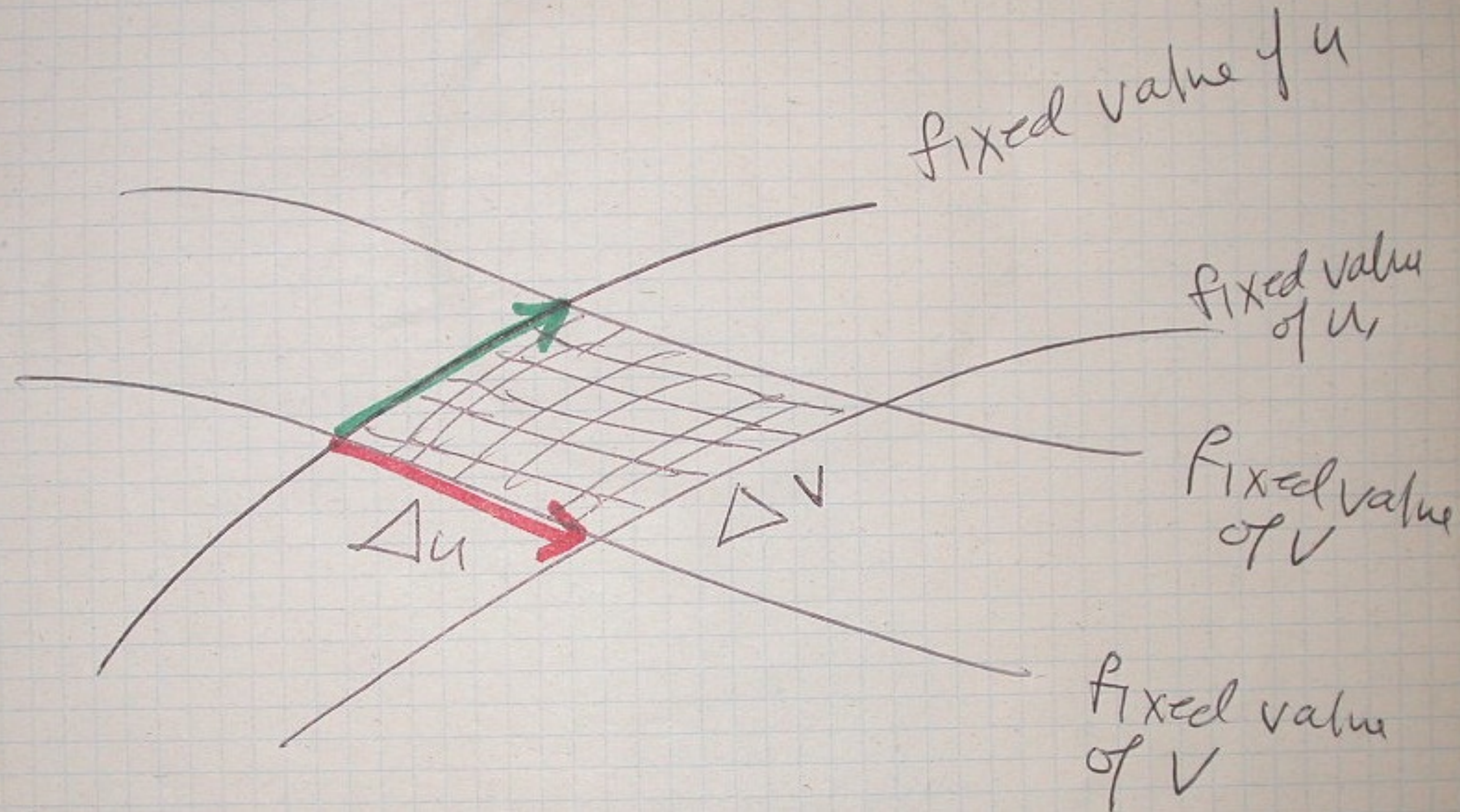


(Adapted from graphic on Wikipedia.)

# *How would we integrate with curvilinear coordinates?*

- Write $x = x(u,v)$, $y = y(u,v)$
  - Example: $x = r \cos \theta$, $y = r \sin \theta$ .
- Make a "differential box" bounded by
  - $(x(u,v),y(u,v))$
  - $(x(u+\Delta u,v),y(u+\Delta u,v))$
  - $(x(u,v+\Delta v),y(u,v\Delta v))$
  - $(x(u+\Delta u,v+\Delta v),y(u+\Delta u,v\Delta v))$


fixed value of u
fixed value of u,
fixed value of v
fixed value of v
Δu
Δv




$\longrightarrow$ is $\cong$ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial u}\Delta u$

$\longrightarrow$ is $\cong$ vector from $X(u,v)$ to $\vec{X}(u,v) + \frac{\partial \vec{X}}{\partial v}\Delta v$

$$\text{Area} \cong \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| \Delta u \Delta v$$

$$=: |J(u,v)| \Delta u \Delta v,$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

sometimes written: $\dfrac{\partial(x,y)}{\partial(u,v)}$

# *Check on Jacobi with an example we know:*

- Example: x = r cos θ, y = r sin θ .
  - J(u,v) =

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Therefore dA = r   dr d θ