MATH 2411 - Harrell

Green potatoes

Lecture 24

This week's announcements

- There will be a test next Wednesday

- The final exam has been scheduled for Tuesday, 30 April, 11:30-2:20 in Clough 102.
  - If you have a conflict with this time, inform Prof. H by the end of next week.

Maxwell's equations

Wikipedia article on *Maxwell's Equations*:

General case

[edit]

The Equations are given in SI units. See below for CGS units.

Name	Differential form	Integral form
Gauss's law:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_S}{\epsilon_0}$
Gauss' law for magnetism (absence of magnetic monopoles):	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{d\Phi_{B,S}}{dt}$
Ampère's Circuital Law (with Maxwell's correction):	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \epsilon_0 \frac{d\Phi_{E,S}}{dt}$

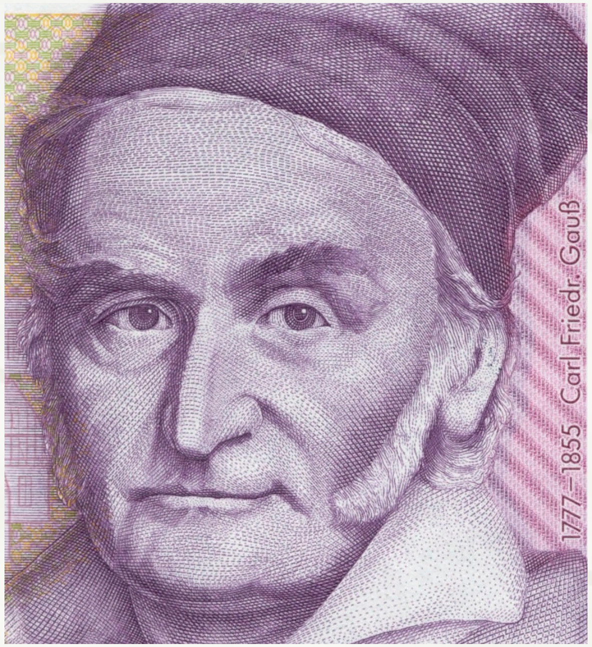
The following table provides the meaning of each symbol and the SI unit of measure:

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\nabla \cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla \times$	the curl operator	

Coming attractions: integrating Grad, Curl, and Div

∇f

Green, Gauß, and Stokes



Meet George Green

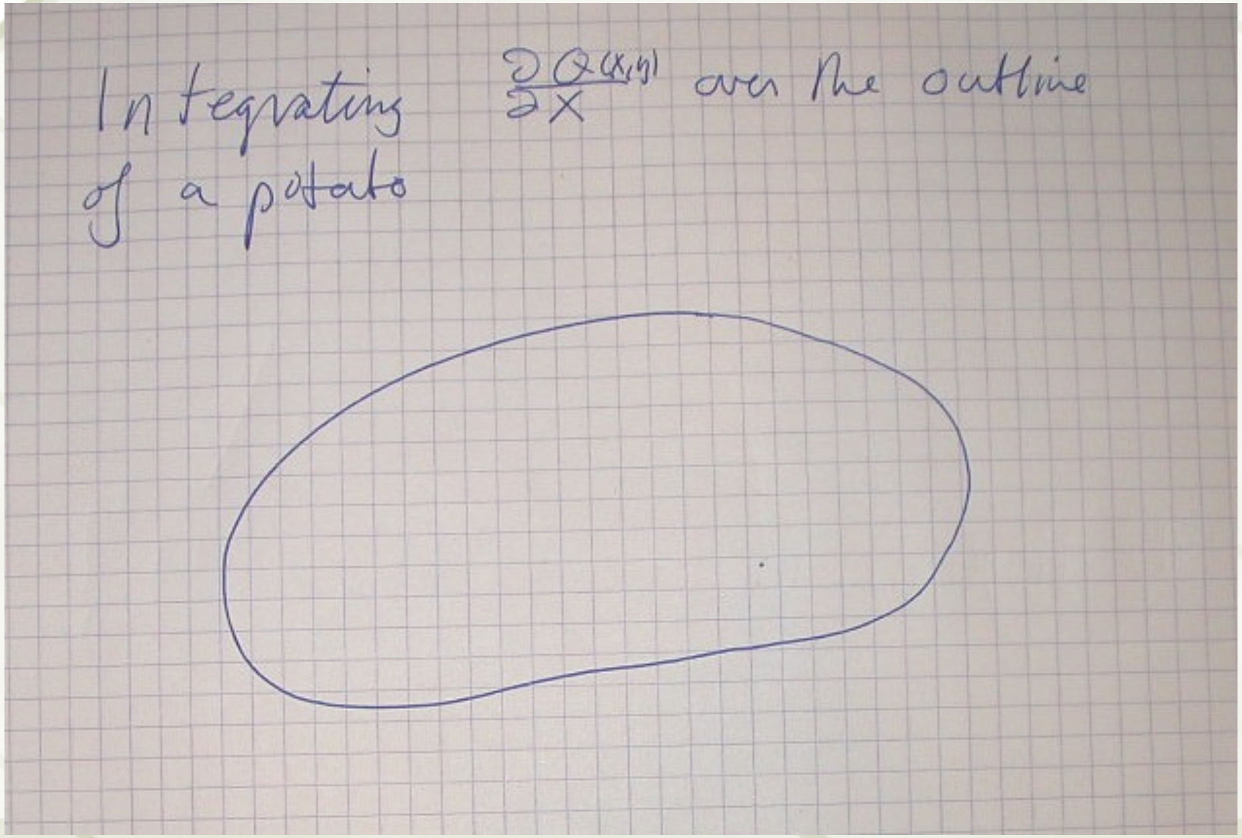
Portrait from Mactutor archive.

The next best thing to the fundamental theorem of calculus for double integrals

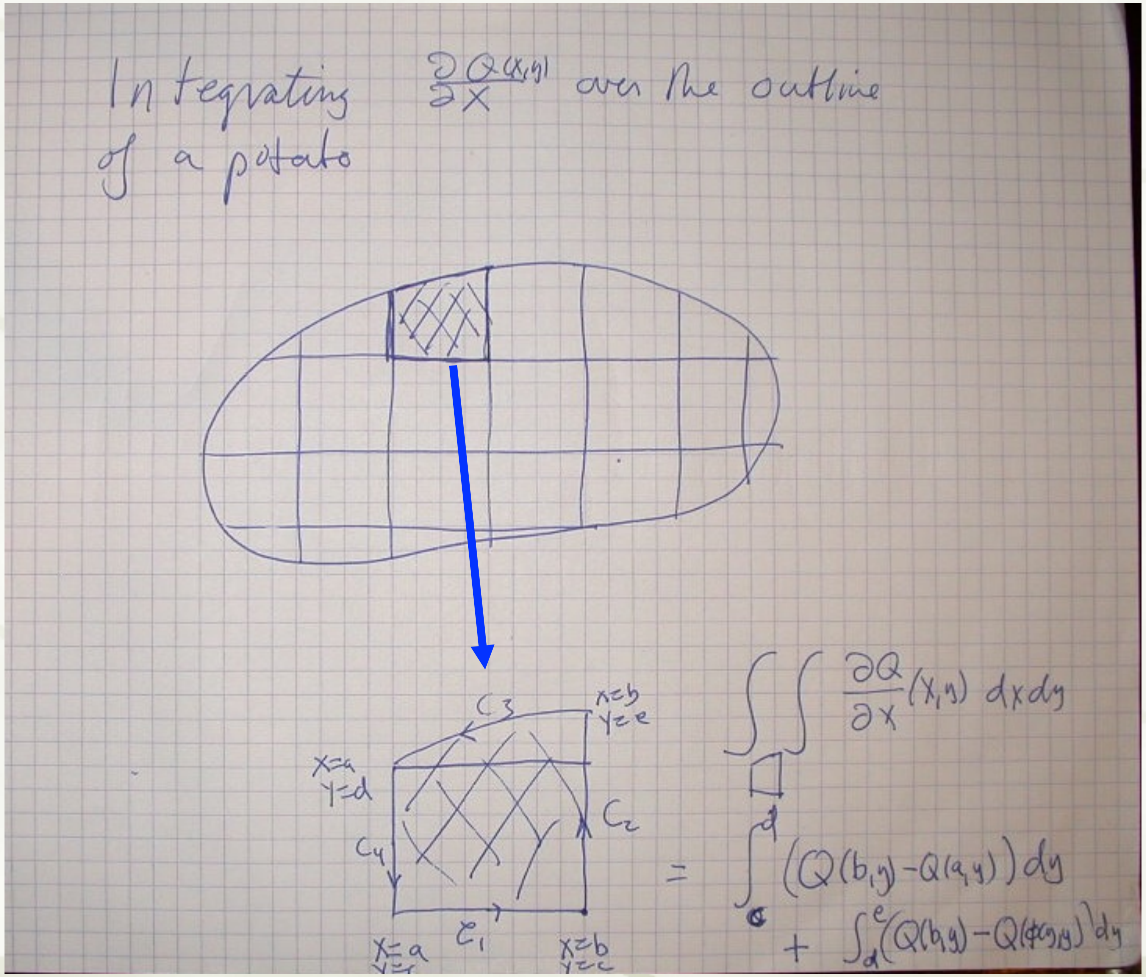
Usual fundamental theorem - an integral can “cancel off” a derivative.

New: If you have a double integral of a derivative, you can cancel one integral off against the derivative. Carefully.

Consider the humble potato, Ω



As usual, we chop the potato up:



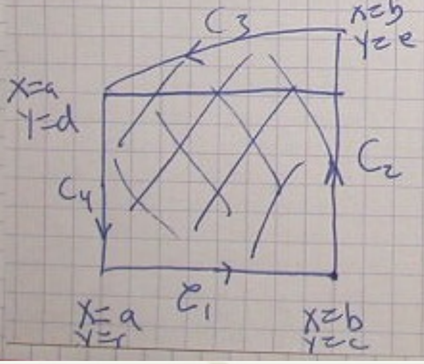
$$\iint_{\square} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

$$\int_c^d \big(Q(b,y)-Q(a,y)\big)\,dy + \int_d^e \big(Q(b,y)-Q(\phi(y),y)\big)\,dy$$

$$= \underset{\substack{C_2\\(\text{up})}}{\int} Q(x,y)\,dy + \underset{\substack{C_4\\(\text{down})}}{\int} Q(x,y)\,dy$$

$$+ \underset{\substack{C_3\\(\text{left-ish})}}{\int} Q(x,y)\,dy + \underset{C_1}{\int} Q(x,y)\,dy$$

because 0!



=

In sum,

$$\int\int \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

$$= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} Q\,\hat{j}\cdot d\vec{r}$$

C.C.W

Likewise…

$$\int_{\Omega_N}\int \frac{\partial P}{\partial y}(x,y)\,dx\,dy = -\int_{\partial\Omega_N} P(x,y)\,dx$$

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$$\int\limits_{\Omega_N}\int \frac{\partial P}{\partial y}(x,y)\,dx\,dy = -\int\limits_{\partial\Omega_N} P(x,y)\,dx$$

$$\oint(-P\hat{i}\,)\cdot d\vec{r}$$

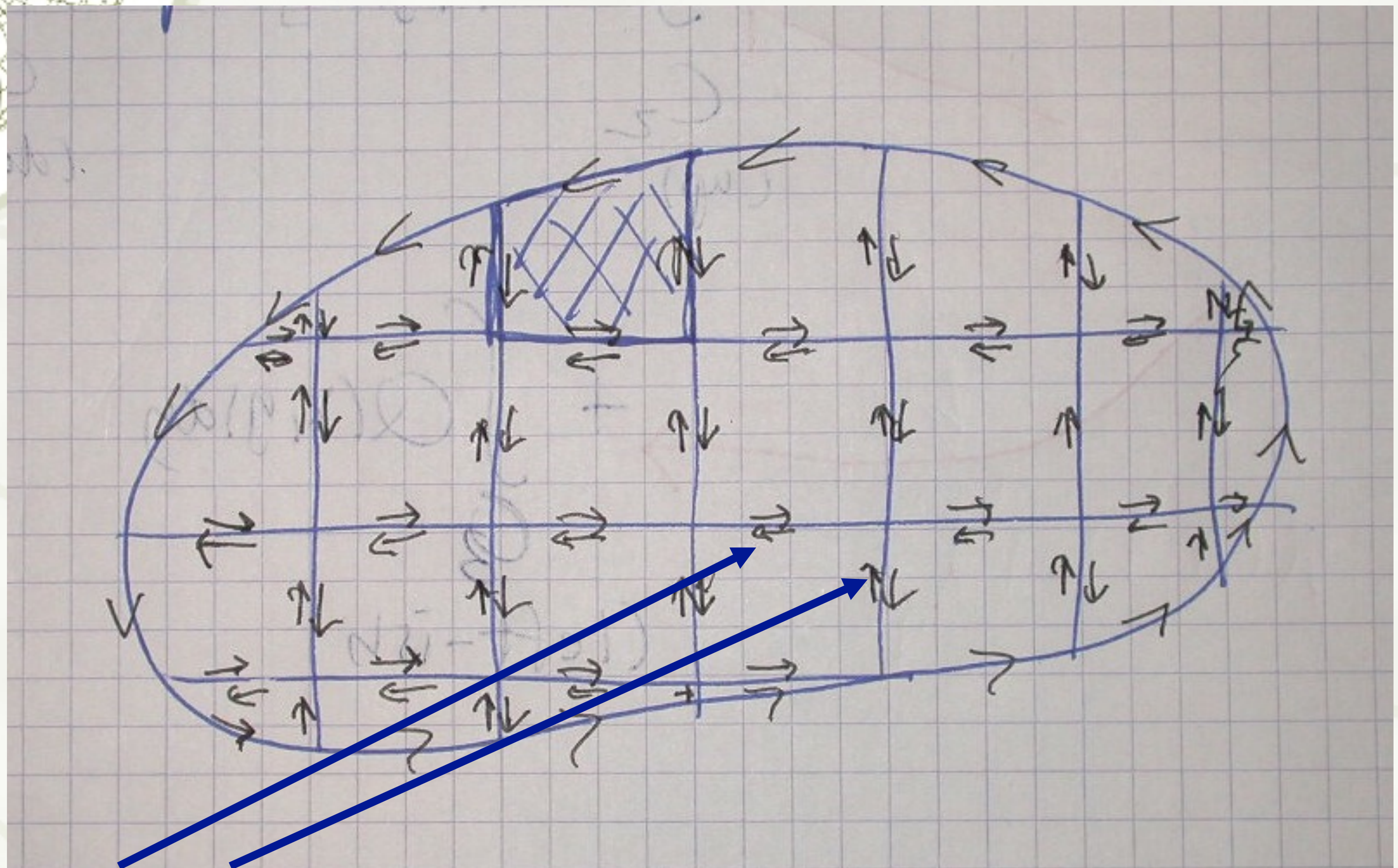
$$d\vec{r} = d\vec{s} = \hat{i}\,dx + \hat{j}\,dy$$

In sum,

- The counterclockwise integral of P **i** around the edge of a little sort-of rectangle is a double integral of – P_y. over the sort-of rectangle.
- The counterclockwise integral of Q **j** around the edge of a little sort-of rectangle is a double integral of Q_x. over the sort-of rectangle.

- So… there is a formula for the line integral of **F**•d**r** = (P **i** +Q **j**)•d**r**.

Now integrate over the whole potato



All interior contributions cancel! Once right, once left. Or once up, once down.

Green's formula

$$\int\int_{\Omega}\left(\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right)dxdy=\oint_C\left(P(x,y)dx+Q(x,y)dy\right)$$

This part is 0 if **F** = ∇ f.

This part is $\oint_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$

Green's formula

$$\int_{\Omega}\int \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dxdy = \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

The spooky thing about Green's theorem is that you can find out something about the inside by integrating around the outside.

Application: The planimeter

Picture by Paul E. Kunkel, with his kind permission.

What does Green tell us when

1. $P = 0, Q = x$?

2. $P = -y, Q = 0$?

3. $P = -y/2, Q = x/2$?

Green is a two-way street

Green is a two-way street

- The line integral may be easier than the area integral.
- The area integral may be easier than the line integral.

Green is a two-way street

- The line integral may be easier than the area integral
  - Example: unit circle,
    - $P = -y\cos(\pi\,(x^2+y^2)^{7/3})$, $Q = x\cos(\pi\,(x^2+y^2)^{7/3})$

      $\partial Q/\partial x - \partial P/\partial y = 2\cos(\pi\,(x^2+y^2)^{7/3})$.

  - Not so nice inside the circle, but…

Green is a two-way street

- The line integral may be easier than the area integral

- The area integral may be easier than the line integral
  - $\oint (3x^2 + y)\,dx + (2x + y^3)\,dy$

Green is a two-way street

- The line integral may be easier than the area integral

- The area integral may be easier than the line integral
  - $\oint(e^x \cos(2\,y) + \sin(x^4) - 2\,y)\,dx + \oint(y^2 \sin(y^2) \sinh(y^4) - 2\,e^x \sin(2\,y))\,dy$

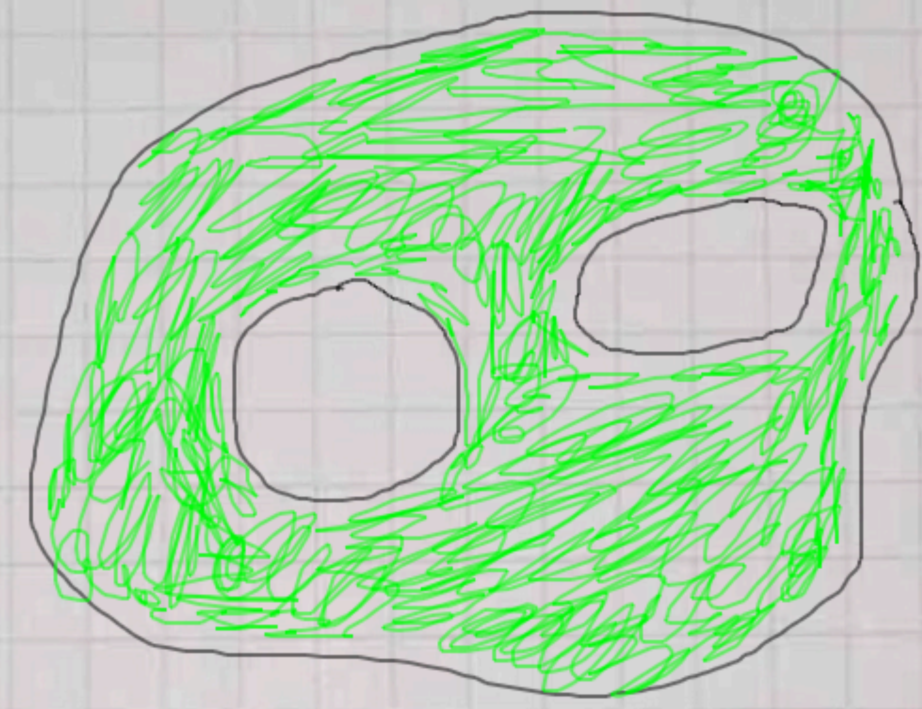
$$P = -\,y \cos(\pi\,(x^2+y^2)^{7/3}),\quad Q = x \cos(\pi\,(x^2+y^2)^{7/3})$$

$$\partial Q/\partial x - \partial P/\partial y = 2 \cos(\pi\,(x^2+y^2)^{7/3}).$$

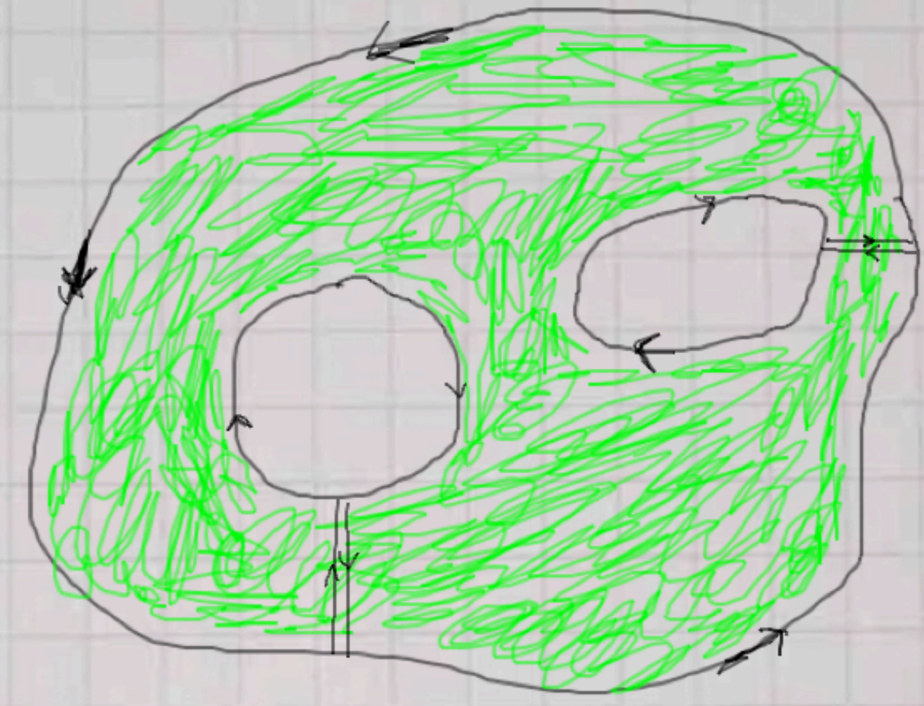
$$\iint_{\{x^2+y^2\le 1\}} 2\cos(\pi\,(x^2+y^2)^{7/3})$$

$$= \oint y\,dx \mp x\,dy$$

Holey Green regions, Batman

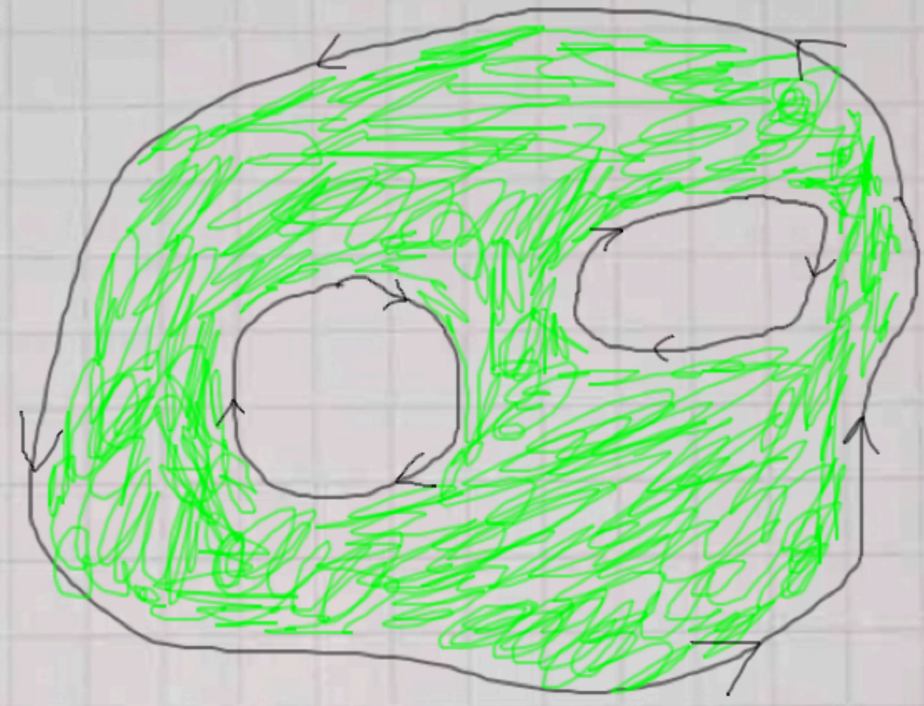


Holey Green regions



On the inside, counterclockwise is clockwise!

Holey Green regions

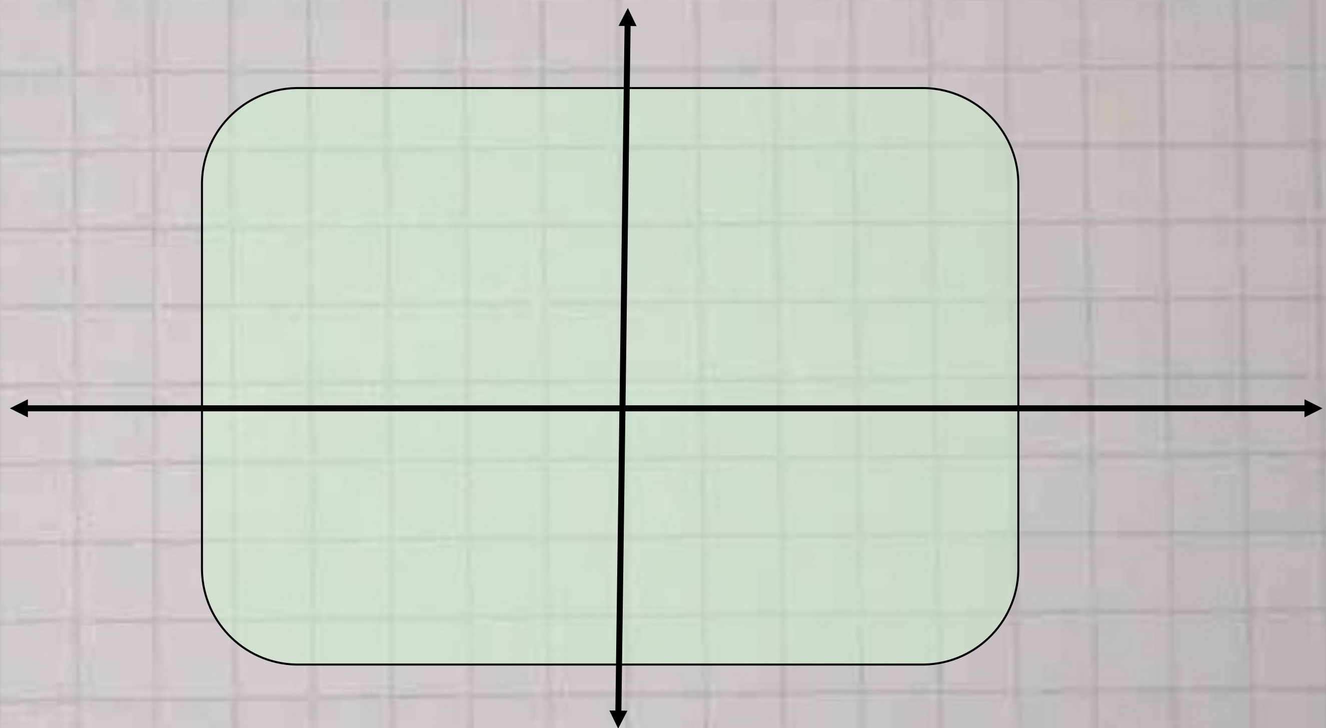


31. Let C be a piecewise-smooth Jordan curve that does not pass through the origin. Evaluate

$$\oint_C \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$

(a) if C does not enclose the origin.

(b) if C does enclose the origin.



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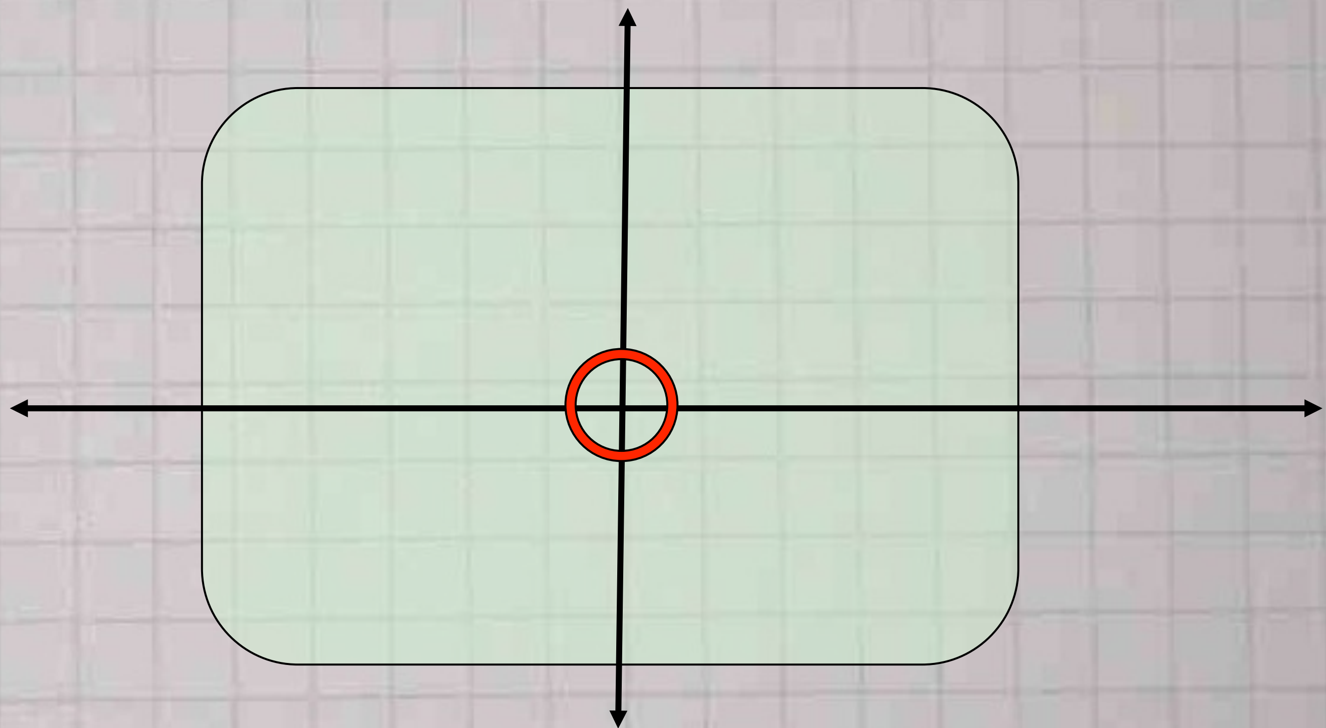

The vector form of Green

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$$

But remembering...

- There will be a test next Wednesday 
- What would you like to review now and next week?

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1 rel

$$\boxed{4y^2+z^2=1+x^2}$$

3 var

$\Rightarrow (3-1)=2$ dim.

≥ 0 rt. 1/2 only

Solve for $x = \sqrt{-1+4y^2+z^2}$

$= \sqrt{4r^2\sin^2\theta + r^2\cos^2\theta - 1}$

sin

think of twisted cyl coords

(r, θ)

x

$y = r\cos\theta$

$z = r\sin\theta$