MATH 2411 - Harrell

Divergent thinking

Lecture 28

Announcements and so forth

- CIOS forms have opened
  - I do look at them! (And they send me reports about how many have filled 'em in.)

The divergence theorem

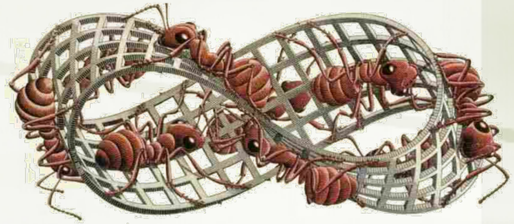
- A.K.A. Gauß’s theorem, but might be due to Ostragradskii or Green, or even Lagrange.
- We’ll get there, but first we will diverge from the topic.

Let's talk just a little more about the curl.

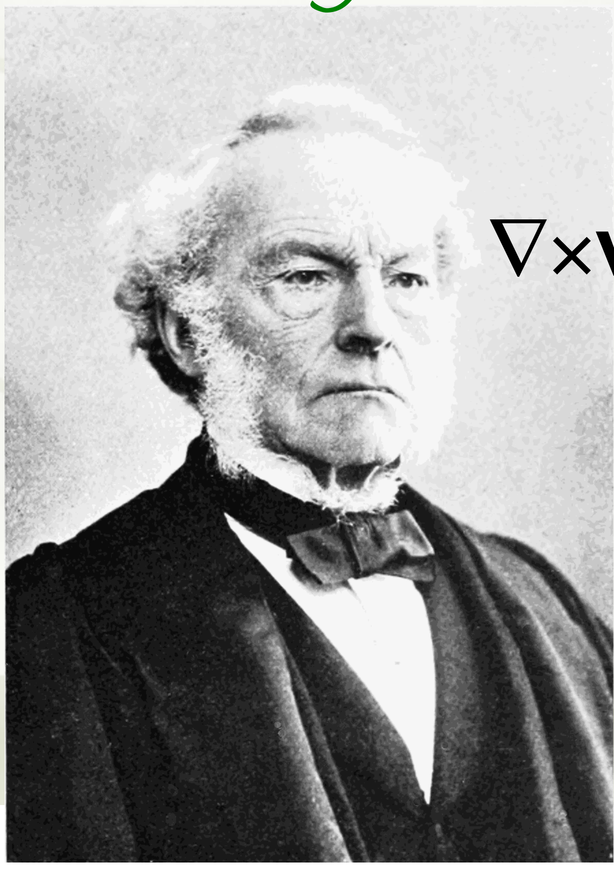
- When we looked at plots of vector fields, we got the impression that a positive curl (say for the **k** component) indicated a counterclockwise circulation about the vector **k**.
- *Stokes now allows us to quantify this and relate it to the "circulation" of the vector field around a given curve.*


me want
sushi!!
ゴジラ!!
What happens when....?
∇×v• n

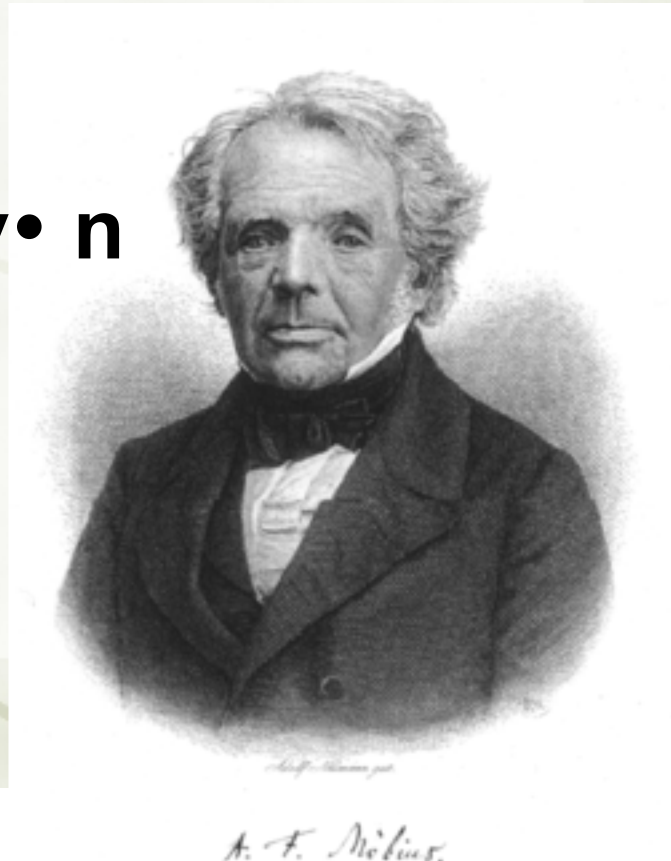
What happens when Stokes meets Möbius?



- Just curious.
  It's a math thing.



$\nabla \times \mathbf{v} \bullet \mathbf{n}$



THE MÖBIUS STRIP AND STOKES' THEOREM

(Thx to Spiro Karigiannis of the University of Waterloo!)

1. Stokes' Theorem

Let us recall Stokes' Theorem:

Theorem 1.1. *Let M be an oriented surface in $\mathbb{R}^3$ with boundary given by the closed curve γ, with orientation induced from that of M (by the right hand rule.) Let $\mathbf{F}(x, y, z)$ be a vector field. Then*

$$\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_\gamma \mathbf{F} \cdot d\mathbf{r} \tag{1.1}$$

In words, the flux of the *curl* of the vector field $\mathbf{F}$ through the surface M in the direction of $\mathbf{n}$ is equal to the circulation of the field $\mathbf{F}$ around the boundary curve γ in the associated direction.

2. The Möbius Strip

Let us now describe the *Möbius strip* and try to use Stokes' Theorem on it. The Möbius strip is obtained by taking a rectangular strip of paper, and gluing two sides together after performing a twist. Let's be more precise: First, imagine constructing a surface of revolution, by taking a curve in the y-z plane and rotating it about the z-axis. Our curve will be the straight line segment $x = R$, $-L \leq y \leq L$, for some positive constants R and L. In this case we just get a cylinder as in Figure 1:

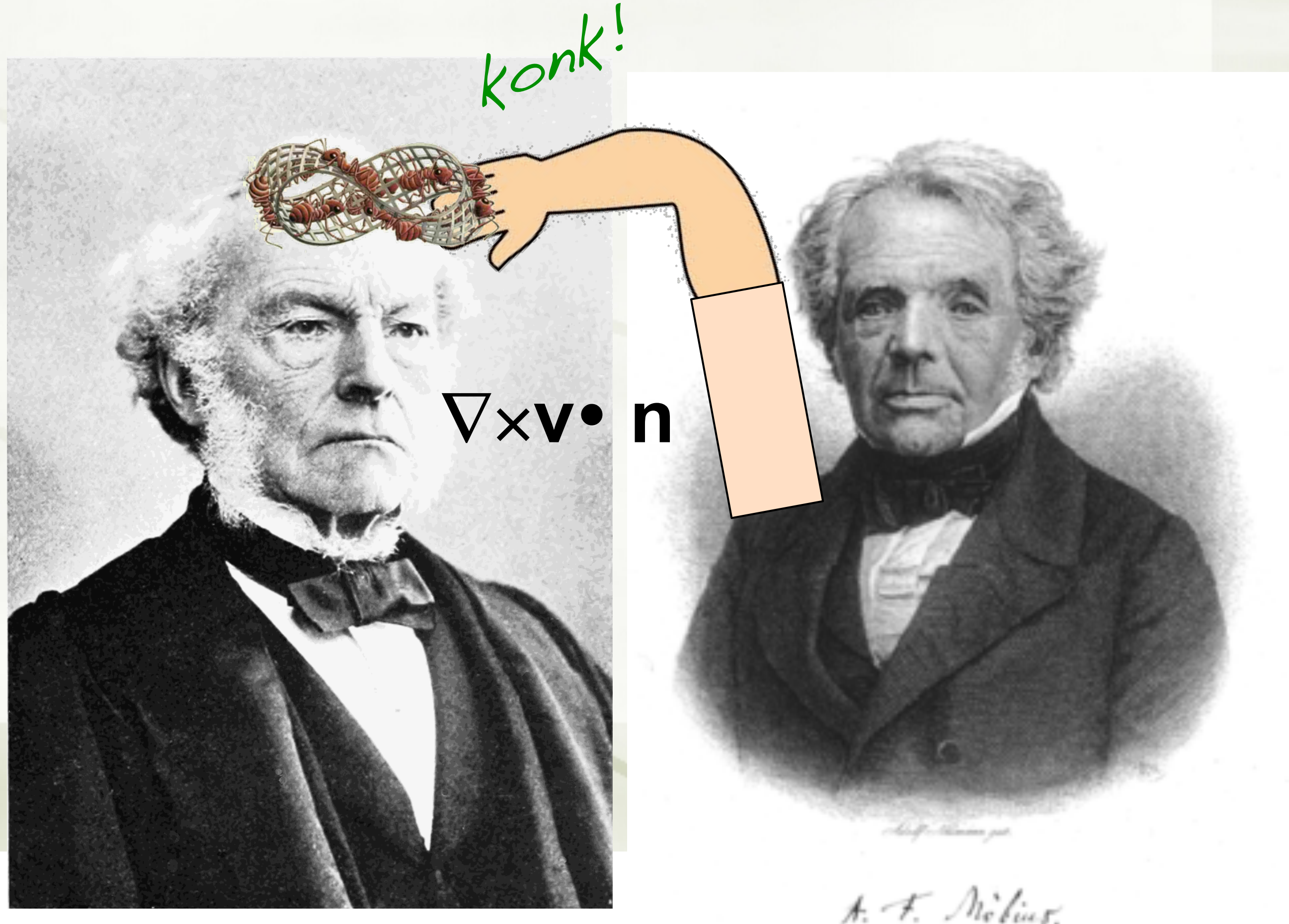
Figure 1. Cylinder

See the rest at

http://www.math.uwaterloo.ca/~karigian/teaching/multivariable-calculus/moebius.pdf

One side of Stokes's theorem evaluates to 4π, and the other side to 0.

What happens when Stokes meets Möbius?



Next topic:

OK, so a static magnetic field is the curl of something. When is that possible?

OK, so a static magnetic field is the curl of something. When is that possible

- Well, a v.f. is a gradient if $\nabla\times\mathbf{F} = \mathbf{0}$...

  and it is a curl if $\nabla\bullet\mathbf{F} = 0$.

Cool! Can you prove that?

We are looking for a $\mathbf{G}$ so that $\mathbf{F} = \nabla\times\mathbf{G}$. But there is more than one answer, because if I change

$$\mathbf{G} \to \mathbf{G} + \nabla f = \mathbf{G}', \text{ then } \nabla\times\mathbf{G} = \nabla\times\mathbf{G}'.$$

So I can make some special choices. Like maybe $G_3 = 0$.

Claim: If $\nabla\cdot\vec{F} = 0$ (Ansatz)

Then a tricky-enough person can find $\vec{G}$ st $F = \nabla \times G$

Hope $\vec{G} = G_1\hat{\imath} + G_2\hat{\jmath} + 0\cdot\hat{k}$

calculate.

Plugged into eqn.

$$\boxed{F = \nabla \times G}$$

$$G_2(x,y,z) = G_2(x,y,0) - \int_0^z F_1(x,y,t)\,dt$$

$$G_1(x,y,z) = G_1(x,y,0) + \int_0^z F_2(x,y,t)\,dt$$

also have k cpt.

$$F_3(x,y,z) = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial G_2(x,y,0)}{\partial x} - \int_0^z \frac{\partial F_1}{\partial x}(x,y,t)\,dt - \frac{\partial G_1}{\partial y}(x,y,0) - \int_0^z \frac{\partial F_2}{\partial y}(x,y,t)\,dt = \frac{\partial G_2(x,y,0)}{\partial x} - \frac{\partial G_1}{\partial y}(x,y,0) + \int_0^z \frac{\partial F_3}{\partial z}(x,y,t)\,dt = \text{Int cst} + F_3(x,y,z)$$

$$0 = \nabla\cdot F = \boxed{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

So, would it be a dumb question to ask whether a function is a divergence if some other derivative is 0?

- Weeeellll……It's not your best question.

The divergence theorem in 3-D.
(Gauß's theorem)

2 D : $$\iint_{\Omega} \nabla \cdot H \, d^2x = \oint_{\partial\Omega} (\vec{H} \cdot \hat{n}) \, ds$$

3 D : $$\iiint_{\Omega} \nabla \cdot H \, d^3x = \iint_{\partial\Omega} (\vec{H} \cdot \hat{n}) \, d^2x$$

n D : $$\int_{\Omega} \cdots \int \nabla \cdot H \, d^n x = \int_{\partial\Omega} \cdots \int (\vec{H} \cdot \hat{n}) \, d^{n-1} x$$

Divergent examples

- Find the flux of **v** = $xy\,\mathbf{i} + x^2y^2\,\mathbf{j}$ across the quarter ellipse $x = 4\cos t$, $y = \sin t$, $t = 0..\pi/2$?
- A fluid has velocity

  $(xy^3 - \sin(y))\mathbf{i} + (yx^3 - \cos(y))\mathbf{j}$.

  What is the rate of flow out of the rectangle $0 \le x \le 3$, $0 \le y \le 2$?

Other divergent thoughts.
It’s kind of warm in here, isn’t it?

Maxwell's equations

Wikipedia article on *Maxwell's Equations*:

General case

[edit]

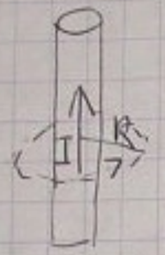
The Equations are given in SI units. See below for CGS units.

Name	Differential form	Integral form
Gauss's law:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_S}{\epsilon_0}$
Gauss' law for magnetism (absence of magnetic monopoles):	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Faraday's law of induction:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{d\Phi_{B,S}}{dt}$
Ampère's Circuital Law (with Maxwell's correction):	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \epsilon_0 \frac{d\Phi_{E,S}}{dt}$

The following table provides the meaning of each symbol and the SI unit of measure:

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\nabla \cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla \times$	the curl operator	

Magnetic field near a wire



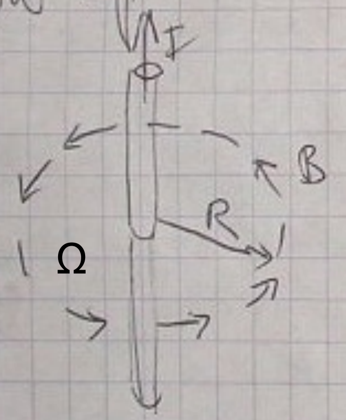
Ampère's law

$\nabla \times B = \mu_0 J$

$$\Longrightarrow \int_{\Omega} (\nabla \times B) \cdot n = \mu_0 \int_{\Omega} J \cdot \hat{n} = \mu_0 I$$

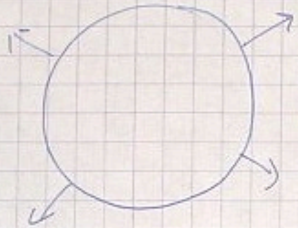
$$\text{also} = \int_{\partial \Omega} \vec{B} \cdot d\vec{r} = 2\pi R \, |B|$$

Blow up



$$\text{so} \ldots \quad |B| = \frac{\mu_0 I}{2\pi R}$$

Suppose a charge density ρ is distributed inside a sphere independently of angle.
What is the electric field $\vec{D}$ at the boundary?



(In free space **E** and **D** are the same.)

S1. Symmetry indicates that it will (a) point radially and (b) have a constant magnitude

Therefore $\int_{\partial B} \vec{E}\cdot\hat{n}\,da = |E|\cdot 4\pi R^2$

area of sphere.

By Gauß, $\int_{\partial B} \vec{E}\cdot\hat{n}\,da = \int_B \nabla\cdot E\,d^3x$

$$= \frac{1}{\varepsilon_0}\int \rho\,d^3x = \frac{Q}{\varepsilon_0}.$$

Conclusion: $\boxed{|E| = \frac{Q}{4\pi\varepsilon_0 R^2}}$ $\boxed{\vec{E} = \frac{Q\,\vec{r}}{4\pi\varepsilon_0|\vec{r}|^3}}$

Figuring out the volume without getting inside the potato.

The blinding light of discovery through PDEs

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James Clerk Maxwell

From Wikipedia, the free encyclopedia

James Clerk Maxwell FRS FRSE (13 June 1831 – 5 November 1879) was a Scottish theoretical physicist.[1] His most prominent achievement was formulating a set of equations that united previously unrelated observations, experiments, and equations of electricity, magnetism, and optics into a consistent theory.[2] His theory of classical electromagnetism demonstrates that electricity, magnetism and light are all manifestations of the same phenomenon, namely the electromagnetic field. Maxwell's achievements concerning electromagnetism have been called the "second great unification in physics",[3] after the first one realised by Isaac Newton.

Maxwell demonstrated that electric and magnetic fields travel through space in the form of waves at the speed of light in 1865, with the publication of *A Dynamical Theory of the Electromagnetic Field*. Maxwell proposed that light was in fact undulations in the same medium that is the cause of electric and magnetic phenomena.[4] The unification of light and electrical phenomena led to the prediction of the existence of radio waves.

Maxwell also helped develop the Maxwell–Boltzmann distribution, which is a statistical means of describing aspects of the kinetic theory of gases. He is also known for presenting the first durable colour photograph in 1861 and for his foundational work on the rigidity of rod-and-joint frameworks (trusses) like those in many bridges.

His discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics. Many physicists regard Maxwell as the 19th-century

James Clerk Maxwell

James Clerk Maxwell (1831–1879)

Born	13 June 1831 Edinburgh, Scotland, UK

The blinding light of discovery through PDEs

Born in Edinburgh on 13th June 1831, Maxwell showed early signs of curiosity but was nicknamed "daftie" by his fellow pupils at Edinburgh Academy. Nevertheless, he sent his first paper to the Royal Society in Edinburgh at the age of 15 and entered Edinburgh University at age 16. He moved to Cambridge University in 1850 and graduated there in 1854.

Maxwell became professor of natural philosophy at Marischal College Aberdeen in 1856 and in 1857 published a paper establishing that the rings of Saturn were clouds of dust. He moved to a professorial post in London in 1860 and while there, demonstrated colour photography for the first time (using a tartan ribbon). He also explained the movement of molecules in gases.

Returning to Edinburgh in 1865 Maxwell worked on electricity and magnetism, propounding the electromagnetic theory of light and that electricity travels at the speed of light. His equations established that electricity and magnetism are aspects of the same entity - electromagnetism.

He predicted the existence of radio waves in 1865, paving the way for radio, TV and electronics and so can be considered to be the father of electronics. His "Treatise on Electricity and Magnetism" containing the famous Maxwell equations was published in 1873. But it was only in 1887 when Heinrich Hertz discovered the existence of radio waves that his calculations became accepted. Nowadays, the Encyclopaedia Britannica describes his Treatise as "one of the most splendid monuments ever raised by the genius of one man."

The blinding light of discovery through PDEs

$$\nabla\times\nabla\times\mathbf{G} = \nabla(\nabla\bullet\mathbf{G}) - \nabla^2\mathbf{G}$$