MATH 2411 - Harrell

# *Tangent vectors, or...*

## *how to go straight when you are on a bender.*

Lecture 3

# *This week's learning plan*

- You will understand curves as functions from a scalar (time?) into space.
- You will be able to calculate the length of a curve.
- You will know about tangent vectors to a curve. Then, you will be able to attach a whole vector frame to the curve
- We'll (probably) start thinking about functions where vectors go in and scalars come out.

# *Reminders and Clarifications*

- Please hand your homework in to Shane on Mondays, rather than trying to submit on T-Square
- We are jumping ahead to curves (§2.4) before discussing partial derivatives.
- Register your clicker! On Thursday you may lose points if you haven’t done that.
- Thursday office period. 11:00-11:20 is already taken, so better to drop in at 11:30 or so. Also, that office period will end sharply at 11:55. If this doesn’t work for you, write me.

# *In our previous episode:*

1. *Vector functions are curves.*

2. *Don't worry too much about the basic rules of calculus for vector functions. They are pretty much like the ones you know and love.*

# Clicker quiz

At time t=1, what is the velocity of an object moving on the path

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - (t-1)^2\,\mathbf{k}\ ?$$

A $\mathbf{i} + \mathbf{j} - 0\,\mathbf{k}$

B $5^{1/2}$ .... *wrong kind of animal!*

C $\mathbf{i} + 2\,\mathbf{j} - 0\,\mathbf{k}$ √

D $\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$

E $2^{1/2}$ .... *wrong kind of animal!*

F none of the above

# *Clicker quiz*

Let $\mathbf{r}(s) = s\,\mathbf{i} + s^2\,\mathbf{j} - (s-1)^2\,\mathbf{k}$ and

$s = \cos(2\pi t)$ (notice, t=1 implies s=1)

Use the chain rule to calculate

$d\mathbf{r}(\cos(2\pi t))/dt$ at $t=1$

A $\mathbf{i} + 2\,\mathbf{j} - 0\,\mathbf{k}$

B $2\pi(\mathbf{i} + 2\,\mathbf{j})$

C 0 .... *wrong kind of animal!*

D **0** √

E $2\pi(\mathbf{i} + 1\,\mathbf{j} - 0\,\mathbf{k})$

F none of the above

# *Now for the fun ... Curves!*

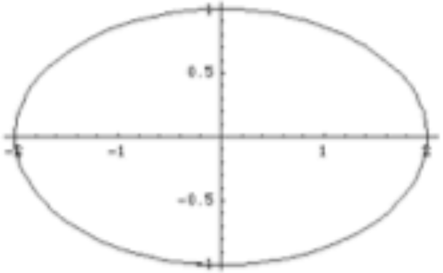
- plane curves
  - circles and ellipses
  - spirals
  - Lissajous figures
- space curves
  - helix

# *How to talk about curves*

- One possibility is to use a parameter.
- Think of it as time, but it doesn't have to literally be time.
- scalar (time) in, vector (space) out

## Some Nice Curves

```
ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```

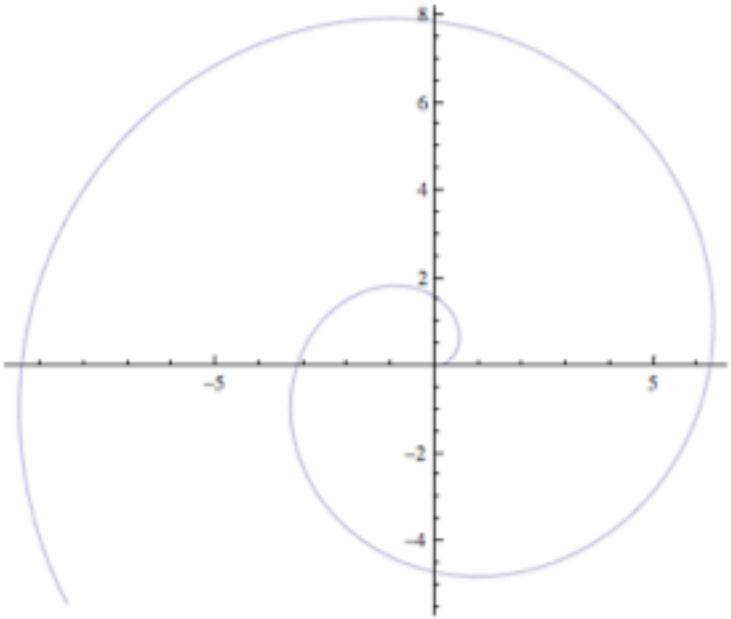



- Graphics -

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```

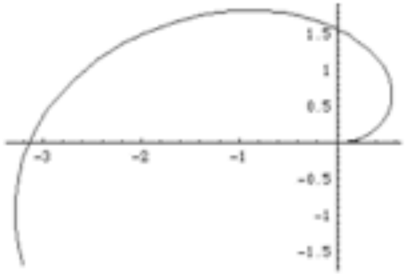



- What is we change the parameter t -> Sinh[s], for instance?

```
ParametricPlot[Spiral[Sinh[s]], {s, 0, 2}]

ParametricPlot3D[Spiral3D[t], {t, 0, 30}, ViewPoint → {5, 0, 4}]
```

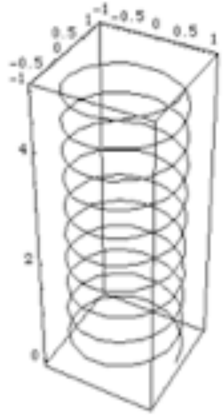
```
ParametricPlot[Spiral[Sinh[s]], {s, 0, 2}]
```



```
- Graphics -
```

```
Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}
```

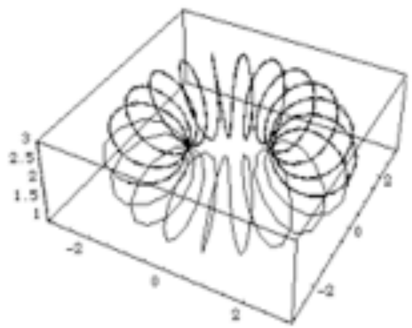
```
ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



```
- Graphics3D -
```

```
Solenoid[t_, r_, R_, w_] := {(R + r Cos[w t]) Cos[t], (R + r Cos[w t]) Sin[t], R + r Sin[w t]}
```

```
ParametricPlot3D[Solenoid[t, 1, 2, 20], {t, 0, 10}, PlotPoints → 360]
```

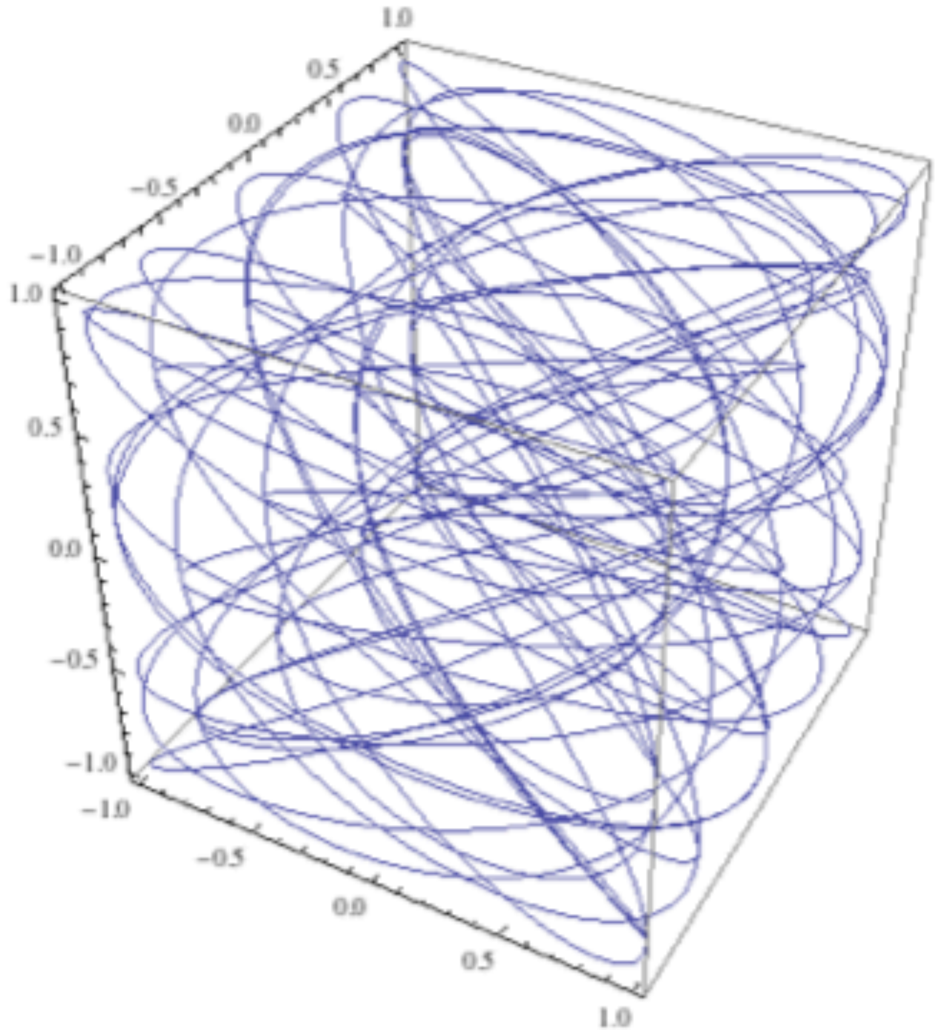


Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

VecSpCurve.nb

In[13]:= `ParametricPlot3D[{Sin[5 t], Cos[3 t], Sin[Pi t]}, {t, 0, 50}]`

Out[13]=




## More Space Curves

100%

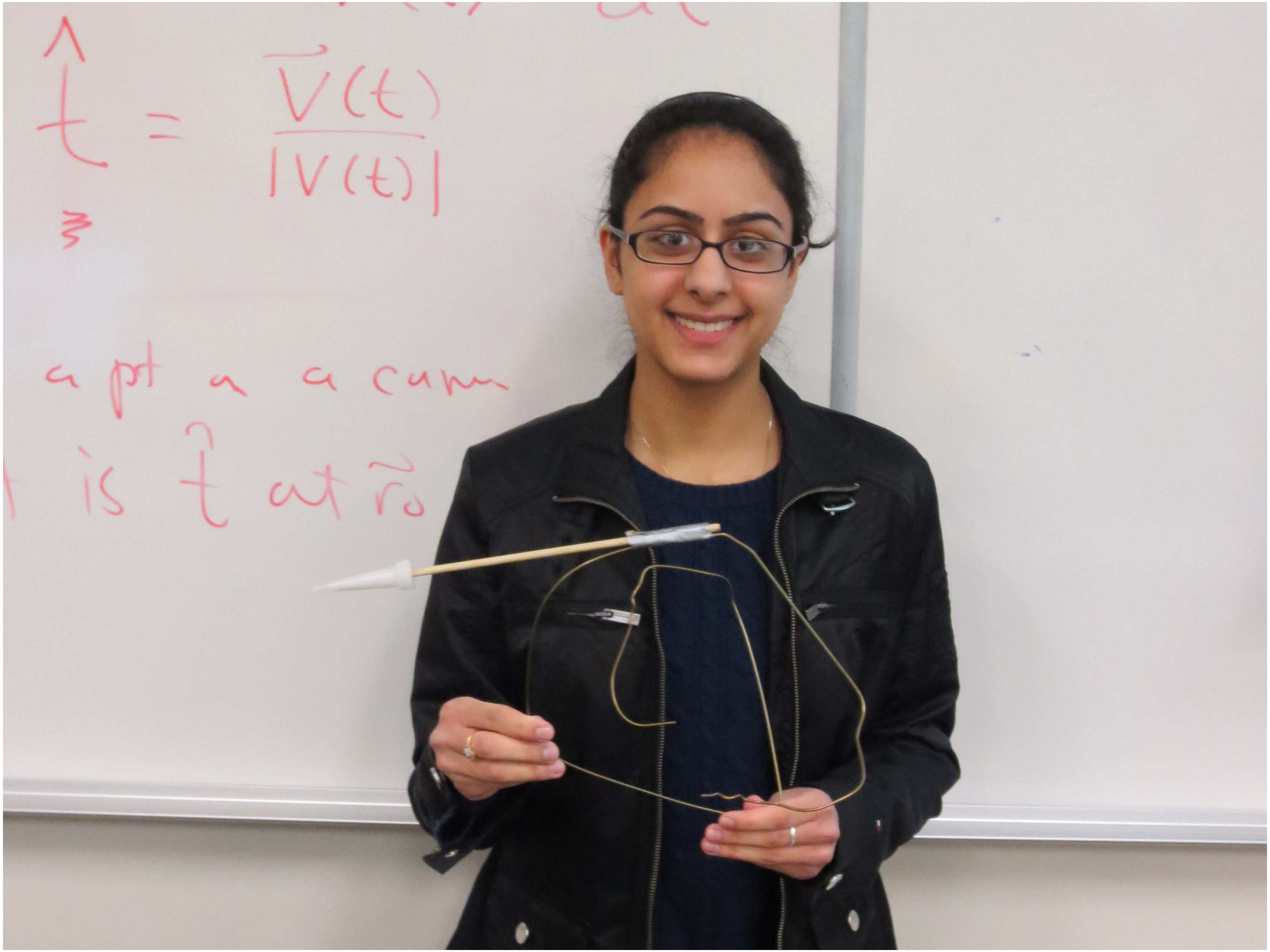
# *Tangent vectors – the derivative of a vector function*

$$\frac{d\mathbf{v}(t)}{dt} = \lim_{h\to 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h}$$

$$= \mathbf{i}\left(\lim_{h\to 0} \frac{v_1(t+h) - v_1(t)}{h}\right) + \mathbf{j}\left(\lim_{h\to 0} \frac{v_2(t+h) - v_2(t)}{h}\right)$$

# *Tangent vectors*

- Think velocity!
- Tangent lines
  - *Tell us more about these!*

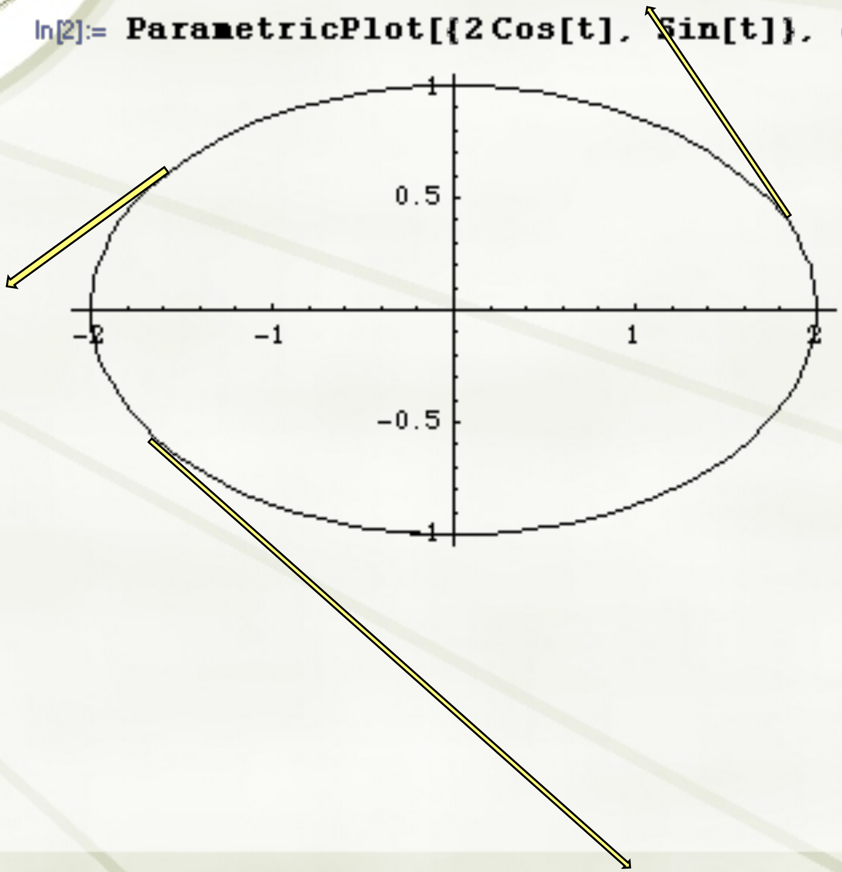

$\hat{t} = \frac{\vec{V}(t)}{|V(t)|}$
is $\hat{t}$ at $\vec{r}_0$

# *Tangent vectors*

- The velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$ is tangent to the curve – points along it and not across it.

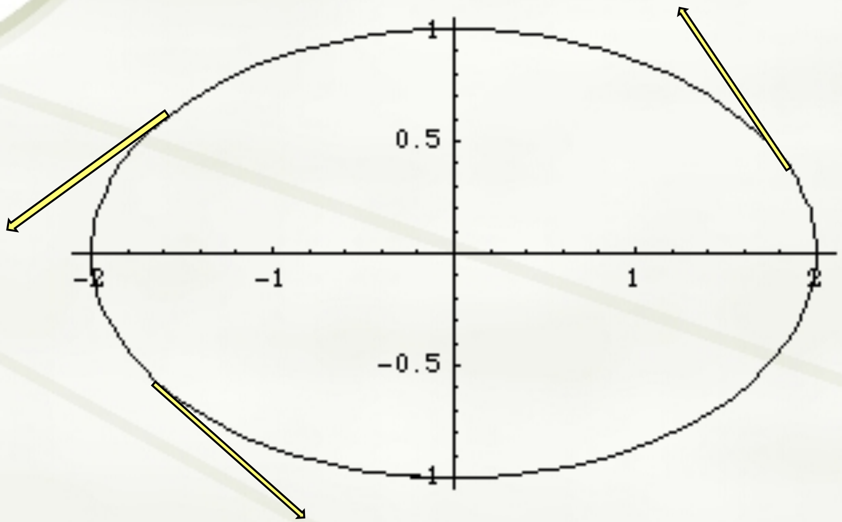
# *Example: ellipse*

In[2]:= `ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]`



# *Example: ellipse*

```
In[2]:= ParametricPlot[{2 Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```



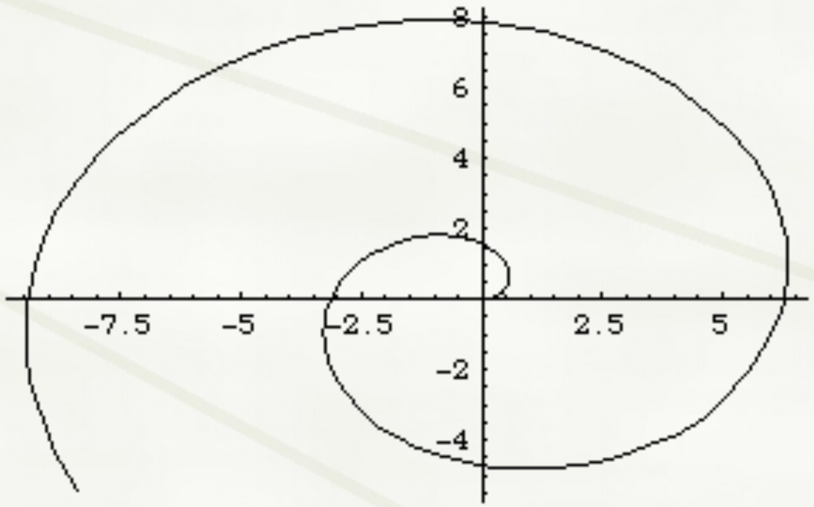

unit tangent vectors – velocity with speed 1

# *Example: spiral*

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



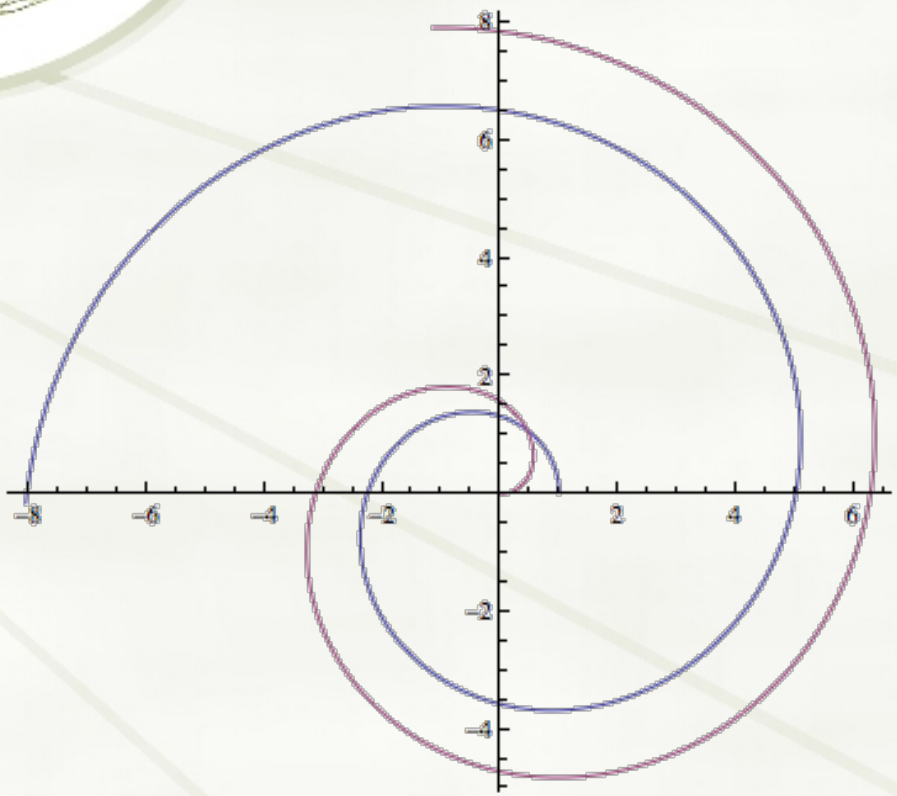
## Differentiating curves

```
D[Spiral[t],t]
```

```
{Cos[t] - t Sin[t], t Cos[t] + Sin[t]}
```

```
ParametricPlot[{%, Spiral[t]}, {t, 0, 8}]
```

(shows curve in space and the curve of velocity)




```
D[Spiral[t], {t, 2}]
```

(2nd derivative)

```
{-t Cos[t] - 2 Sin[t], 2 Cos[t] - t Sin[t]}
```

# Example: spiral

```
Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2 + D[r[[3]], t]^2]

Tang[r_] := {D[r[[1]], t], D[r[[2]], t], D[r[[3]], t]} / Speed[r]

General::spell : Possible spelling error: new symbol name "Tang" is similar to existing symbols {Tan, Tanh}. More…

Tang[Spiral3D[t]]
```

$$\left\{\frac{\mathrm{Cos}[t] - t\,\mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t] + \mathrm{Sin}[t])^2 + (\mathrm{Cos}[t] - t\,\mathrm{Sin}[t])^2}},\ \frac{t\,\mathrm{Cos}[t] + \mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t] + \mathrm{Sin}[t])^2 + (\mathrm{Cos}[t] - t\,\mathrm{Sin}[t])^2}},\ 0\right\}$$

```
Simplify[%]
```

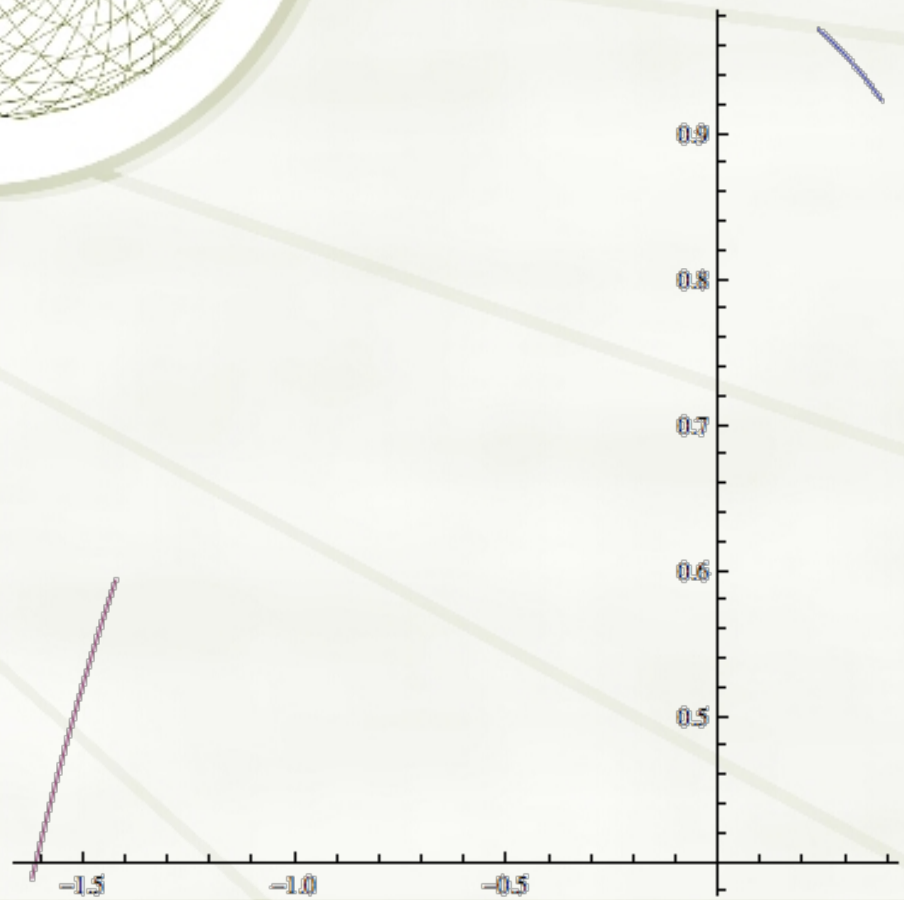
$$\left\{\frac{\mathrm{Cos}[t] - t\,\mathrm{Sin}[t]}{\sqrt{1 + t^2}},\ \frac{t\,\mathrm{Cos}[t] + \mathrm{Sin}[t]}{\sqrt{1 + t^2}},\ 0\right\}$$

# *Some tricky stuff*

- What is the connection between the position and velocity when an object travels in a circle?

In[13]:= 
```
ParametricPlot[{{Cos[Sinh[t]], Sin[Sinh[t]]}, {-Cosh[t] Sin[Sinh[t]], Cos[Sinh[t]] Cosh[t]}},
  {t, 1, 1.1}, AspectRatio → 1]
```

Out[13]=



# *Some tricky stuff*

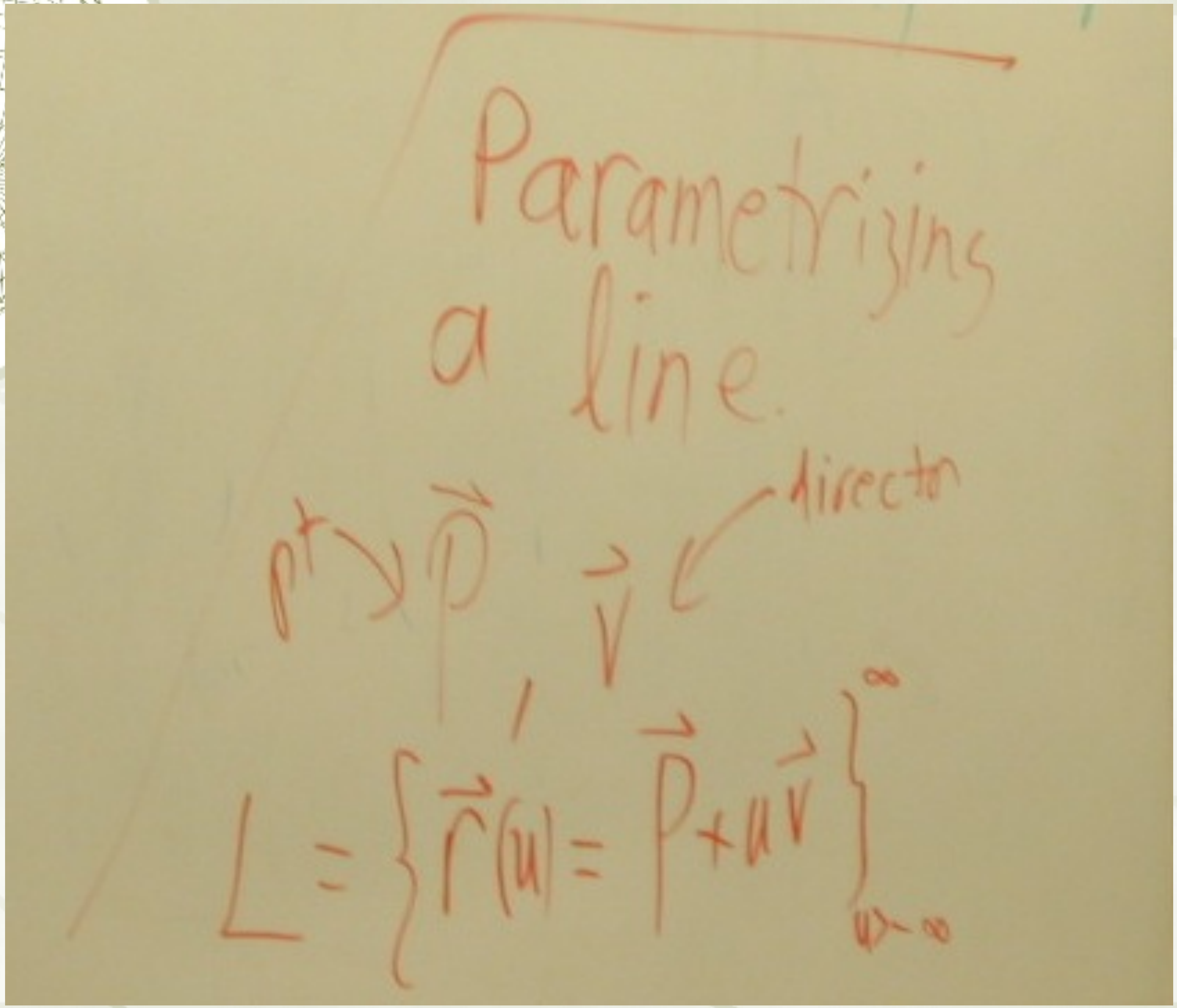
- What is the connection between the position and velocity when an object travels in a circle?
- A:  **r** and **v** are always perpendicular

# *Some tricky stuff*

- What is the connection between the position and velocity when an object travels in a circle?
- A:  **r** and **v** are always perpendicular
  - and vice versa

# *Tangent lines*

- They touch the curve and point along the tangent vector.


Parametrizing a line.
pt → $\vec{p}$ , $\vec{v}$
direction
$L = \{\vec{r}(u) = \vec{p} + u\vec{v}\}_{u=-\infty}^{\infty}$

$$\vec{p} = \begin{bmatrix}3\\2\\1\end{bmatrix}$$

$$\vec{v} = \begin{bmatrix}5\\5\\5\end{bmatrix}$$

$$L = \left\{ \begin{bmatrix}3\\2\\1\end{bmatrix} + u\begin{bmatrix}5\\5\\5\end{bmatrix} \right\}$$

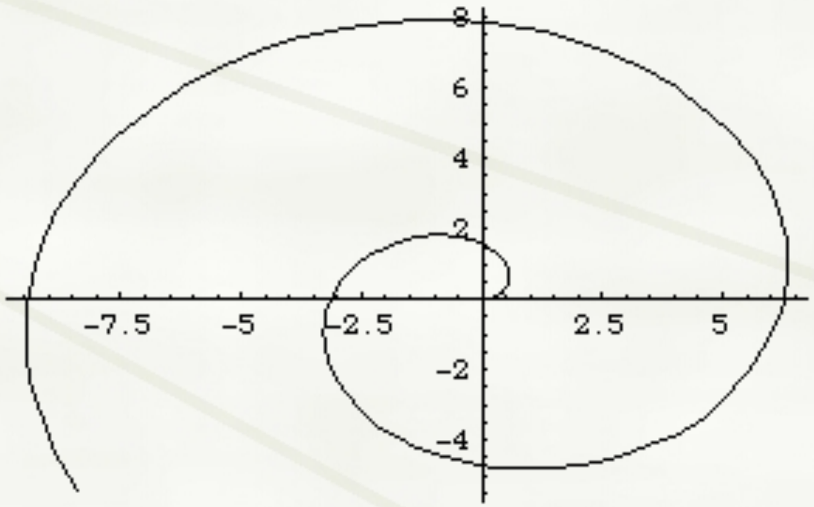
$$= \left\{ \begin{bmatrix}3+5u\\2+5u\\1+5u\end{bmatrix} \right\}$$

# Example: spiral

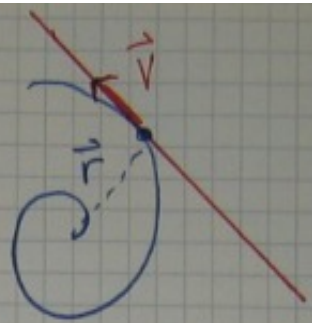
```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Spiral $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}$




Velocity $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{bmatrix}$

This is a tangent vector $\vec{V}$ at pt. $\vec{r}$

$$\boxed{\vec{L} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \vec{r}_0 + u\vec{V} \right\}}$$

* parametric form!

Ex: $t = \frac{\pi}{4}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{bmatrix}$$

$$\vec{V} = \begin{matrix} \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \end{matrix}$$

$$\vec{L} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) \\ \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right) \end{bmatrix}$$

# *Tangent vectors*

- Think velocity!
- Tangent lines
- Approximation and Taylor's formula
- Numerical integration
- Maybe most importantly – **T** is our tool for taking curves apart and understanding their geometry.

# CULTURE BREAK

**Sofía V. Kovalevski (1850-1891):**

**Una brillante Físico-Matemática.**

born on this day, 1850.

Софья Васильевна Ковалевская

*From a presentation by R. Benguria, Santiago de Chile

# Zur Theorie der partiellen Differentialgleichungen *).

(Von Frau *Sophie von Kowalevsky.*)

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## Einleitung.

Es sei eine algebraische Differentialgleichung

$$(1.)\qquad G\left(x, y, \frac{dy}{dx}, \cdots \frac{d^n y}{dx^n}\right) = 0$$

vorgelegt, wo $G$ eine ganze rationale Function der unabhängigen Veränderlichen $x$, der als Function derselben zu bestimmenden Grösse $y$ und der Ableitungen derselben nach $x$ bis zur $n^{\text{ten}}$ Ordnung hin bedeutet.

Artículo de S. Kovalevski (basado en su tesis de doctorado) publicado en el Journal für die reine und angewandte Mathematik vol. 80, pp. 1-32 (1875). [CRELLE's Journal].

En 1884, SK obtiene el derecho a enseñar en la Universidad de Estocolmo. Su primera clase fue el 30 de Enero de 1884.

El mismo año la nombran “Profesor Extraordinarius” por un período de 5 años.

En 1889, fue finalmente nombrada Profesora, y se convierte en la tercera mujer en tener una cátedra en una Universidad Europea.

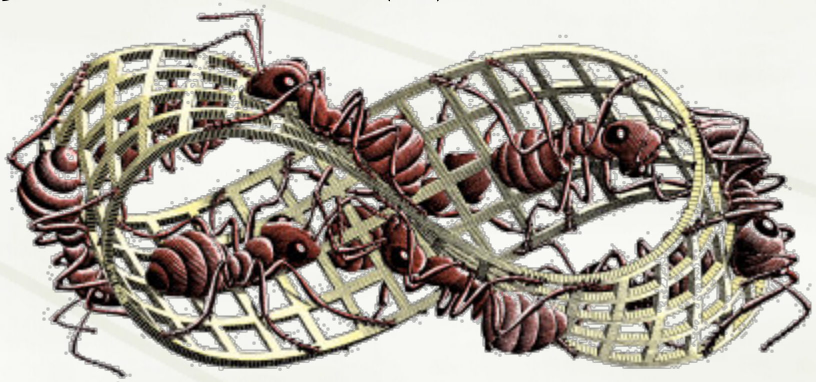
# *Velocity vs. speed*

- The velocity $\mathbf{v}(t) = d\mathbf{r}/dt$ is a vector function.
- The speed $|\mathbf{v}(t)|$ is a scalar function.

  $|\mathbf{v}(t)| \geq 0$.

# *Arc length*

- If an ant crawls at 1 cm/sec along a curve, the time it takes from a to b is the arc length from a to b.
- More generally, $ds = |\mathbf{v}(t)|dt$

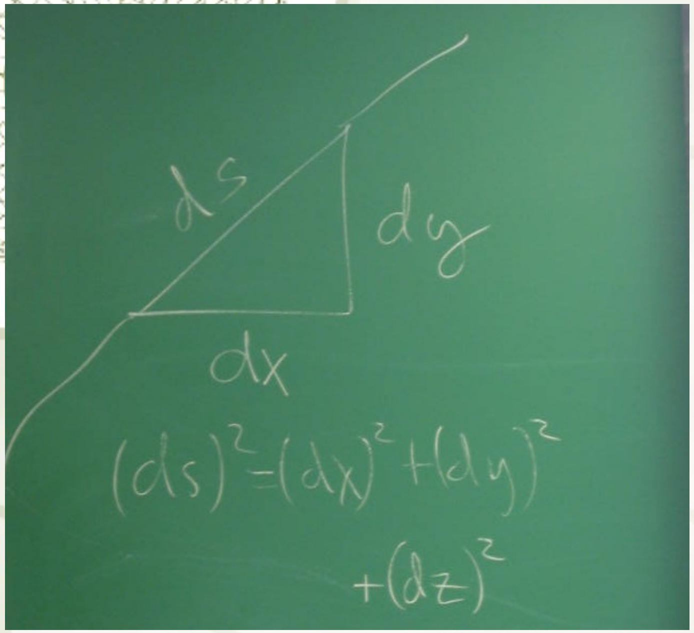


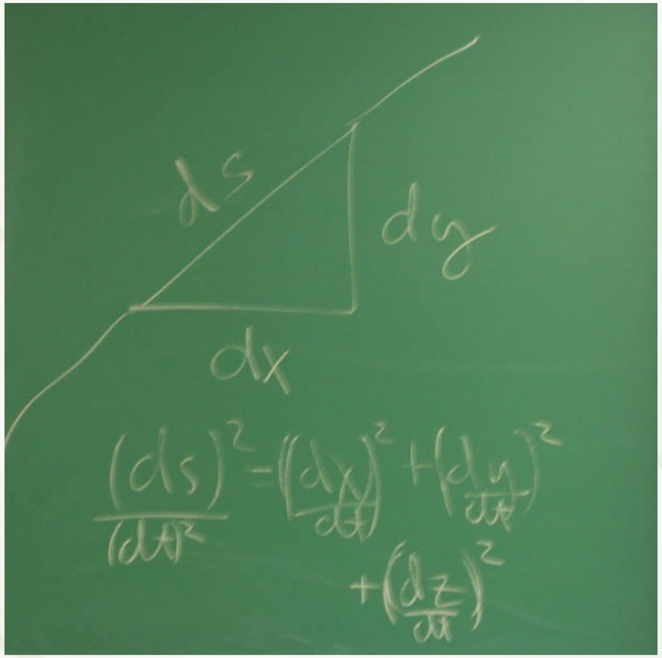
# *Arc length*

- If an ant crawls at 1 cm/sec along a curve, the time it takes from a to b is the arc length from a to b.
- More generally, $ds = |\mathbf{v}(t)|\, dt$
- In 2-D $ds = (1 + y'^2)^{1/2}\, dx$, or

$$ds^2 = dx^2 + dy^2$$

or…


ds
dy
dx
$(ds)^2=(dx)^2+(dy)^2$
$+(dz)^2$


ds
dy
dx
$\frac{(ds)^2}{(dt)^2}=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$
$+\left(\frac{dz}{dt}\right)^2$

# *Arc length*

$$ds = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt$$

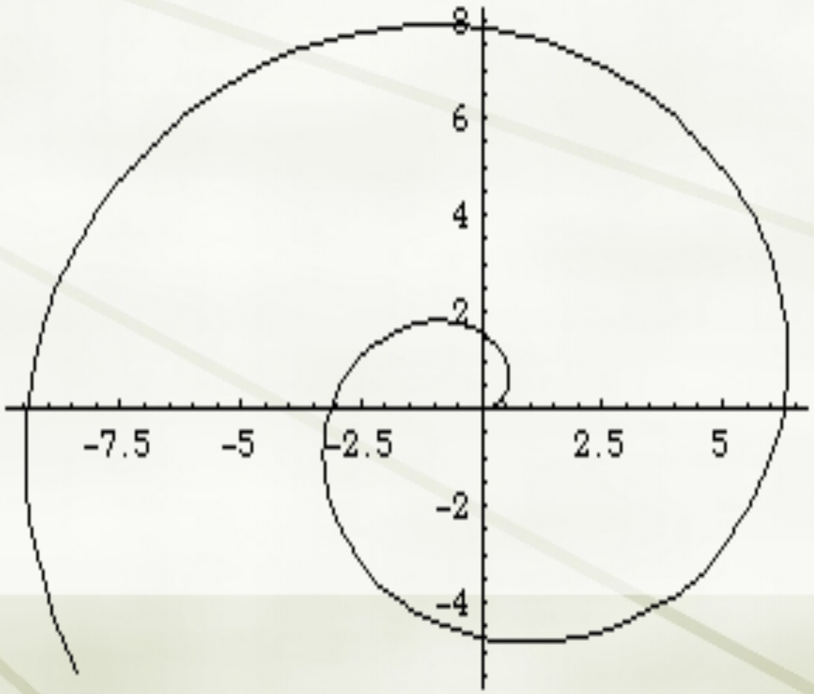
$$L(C) = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt = \int_a^b \|\mathbf{r}'(t)\| dt$$

# Example: spiral

```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



# *Examples*

$$|\vec{r}\,'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t}$$

$$= \sqrt{1+t^2}$$

$$ds = \frac{ds}{dt}\,dt = \sqrt{1+t^2}\;dt$$

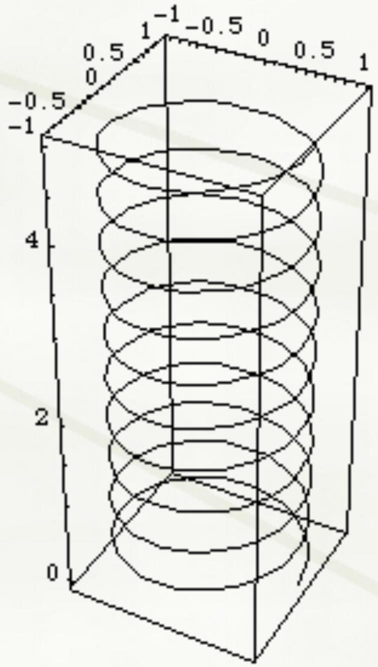
In[1]:= **Integrate[Sqrt[1 + t ^2], {t, 0, 4 Pi}]**

Out[1]= $\frac{1}{2}\left(4\pi\sqrt{1+16\pi^2} + \mathrm{ArcSinh}[4\pi]\right)$

# *Example: helix*

```
In[8]:= Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}

In[9]:= ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



# *Examples*

$$|\vec{r}\,'(t)| = \sqrt{(-4\pi \sin(4\pi t))^2 + (4\pi \cos(4\pi t))^2 + 1}$$

$$= \sqrt{1 + 16\pi^2} \quad \text{(a constant)}$$

$$ds = \sqrt{1+16\pi^2}$$

Miraculously - don’t expect this in other examples - the speed does not depend on t.  The arclength in 2 coils, t from 0 to 1, is the integral of $|r'|$ over this integral, i.e., $(1+ 16\ \pi^2)^{1/2}$.

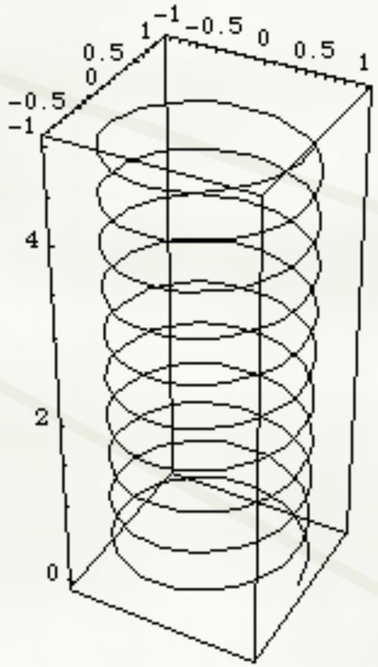
# *Unit tangent vectors*

- Not only useful for arc length, also for understanding the curve 'from the inside.'
- Move on curve with speed 1.
- $\mathbf{T}(t) = \mathbf{r}'(t)/\ |\mathbf{r}'(t)|$

# *Example: helix*

```
In[8]:= Helix[t_] := {Cos[4 Pi t], Sin[4 Pi t], t}

In[9]:= ParametricPlot3D[Helix[t], {t, 0, 5}, PlotPoints → 360]
```



# Example: Helix

Helix

$$\vec{r}(t) = \cos(4\pi t)\hat{\imath} + \sin(4\pi t)\hat{\jmath} + t\hat{k}$$

Velocity

$$\vec{r}\,'(t) = -4\pi\sin(4\pi t)\hat{\imath} + 4\pi\cos(4\pi t)\hat{\jmath} + \hat{k}$$

speed

$$\|\vec{r}\,'(t)\| = \sqrt{(4\pi\sin(4\pi t))^2 + (4\pi\cos(4\pi t))^2 + 1}$$

$$= \sqrt{1+16\pi^2}$$

Velocity changes, as a vector.

Speed does <u>not</u> change.

This is a special fact. Usually

<u>Speed does change</u>

# *Unit tangent vectors*

- Move on curve with speed 1.
- $\mathbf{T}(t) = \mathbf{r}'(t) / |\mathbf{r}'(t)|$
- Only 2 possibilities, $\pm$ **T**.

# Example: Helix

$$\hat{T} = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1+16\pi^2}} \begin{bmatrix} -4\pi \sin(4\pi t) \\ 4\pi \cos(4\pi t) \\ 1 \end{bmatrix}$$

(writing as column vector.)