MATH 2411 - Harrell

Curves from the inside

Lecture 4

Mathematics 2411 **Honors Calculus III** **Spring, 2013**

Current reading and homework assignments

- Due Monday, 20 January

  NOTE: There will be a test on Wednesday, 23 January 

  Reading:
  - MT, Chapter 2, as far as we get by Monday.
  - MT, Section 4.2
  - Lecture of 14 January
  - Optional - the Mathematica notebook that was the source for some calculations in that lecture
  - Lecture of 16 January
  - Chapter 3 and Chapter 4 of Cain-Herod
  - CHECK BACK LATER FOR POSSIBLE ADDITIONAL READINGS.

  Exercises (for test prep, not to be graded):
  - MT (6th ed.), Chapter 1 Review Exercises #1-6
  - MT, Exercises 2.4, #7-14,18,20-23.
  - MT, Exercises 4.2, #4-10, 17-23 (in fact some of these will be done in lecture)..
  - MORE STUDY PROBLEMS FOR THE TEST WILL BE POSTED HERE SOON

  Study break
  - In case you think intersections of lines aren't cool, look at this video

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Current contests

Note about contest entries. These must be entirely your own work and not, for example, copied from the Web, even in modified form, which w

Unless otherwise specified, entries should be submitted to the professor, either in hard copy or by e-mail in a universally readable format, such

- Can you rewrite the grading formula in terms of a data vector, with elements like T1, T@, etc., and vector operations? (The head-scratch

Mathematics 2411 Honors Calculus III Course Description

Spring, 2013 (TTh 3:05 in the Clough Undergraduate Learning Commons, room 102, with recitations MW in Skiles 270)

Instructor: Evans Harrell, Office Skiles 218D, 894 3300, harrell at math.gatech.edu
Instructor's office periods: Monday 11:00-12:00 and Thursday 11:30-12:30 at Skiles 218D. Ordinarily I will also be available for 20 minutes after each class.
Recitation:

Shane Scott

Class web page

The class will be coordinated through T-Square, but you can also consult the Class web page directly. It is your responsibility to consult T-Square or the web page regularly for information about the class, such as homework assignments. You will also be in e-mail contact with the instructor and the teaching assistant, and we will do our best to respond to your questions.

Required texts and materials

We will use Marsden and Tromba, *Vector Calculus*, and on-line materials, which may be linked to from the class website or T-Square. You are also required to have a clicker for classroom use.

Description: Calculus is not only essential in engineering; it is one of mankind's greatest intellectual achievements. After thousands of years of confusion on the part of philosophers, Newton, Leibniz, and Euler created effective conceptual tools for understanding the infinite and the infinitesimal. In the third term of calculus we learn about derivatives and integrals in three (or even more) dimensions and their uses.

Grading and requirements

There will be tests in the recitations on

1. **Wednesday, 23 January**,
2. **Wednesday, 13 February**,
3. **Wednesday, 13 March**,
4. **Wednesday, 10 April**,

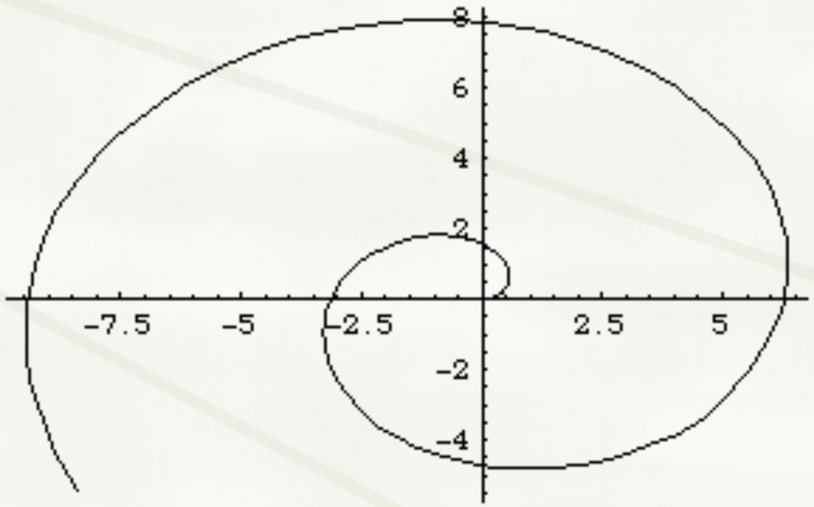
There will also be a final exam, of course. Homework will be collected on Mondays, and there will be regular clicker quizzes. Your homework-quiz average will incorporate at least two drops. In addition, Prof. Harrell may announce occasional contests, and the winner of a contest will receive a small number of extra-credit points.

Example: tangent line to a spiral

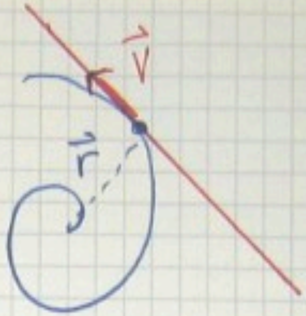
```
Spiral[t_] := {t Cos[t], t Sin[t]}

Spiral3D[t_] := {t Cos[t], t Sin[t], 0}

ParametricPlot[Spiral[t], {t, 0, 10}]
```



Spiral $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}$



Velocity $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{bmatrix}$

This is a tangent vector $\vec{V}$ at pt. $\vec{r}$

$$\boxed{\vec{L} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \vec{r}_0 + u\vec{V} \right\}}$$

* parametric form!

Ex: $t = \frac{\pi}{4}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{bmatrix}$

$\vec{V} = \begin{matrix} \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \end{matrix}$

$\vec{L} = \begin{bmatrix} \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) \\ \frac{\pi}{4\sqrt{2}} + u\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right) \end{bmatrix}$

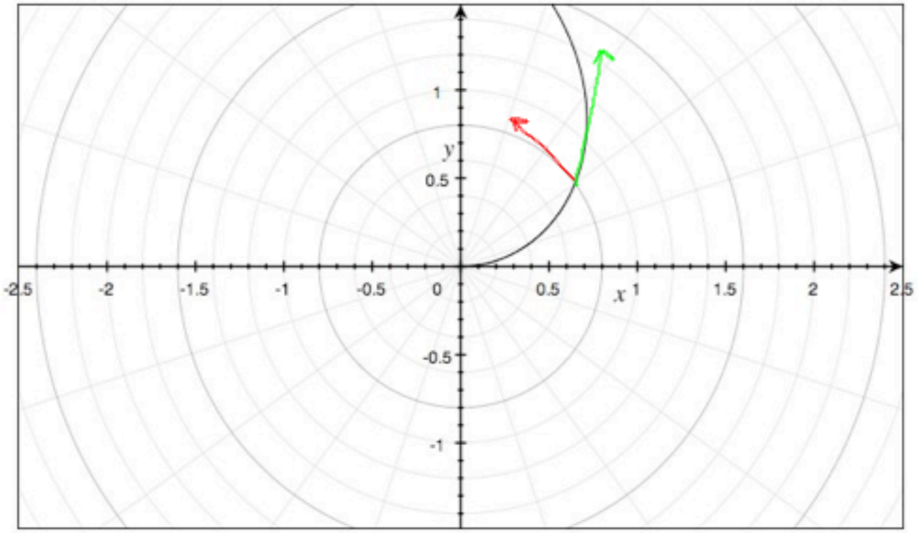
If your careening car smashes into another, how can you calculate the angle of impact?

in your last dying seconds

- The angle between curves is the angle between their tangent vectors. Let’s do an example

Curves

- Angle of intersection
  - Spiral r = 4θ/π and circle



    - $x = 4\,t\cos \pi t$, $y = 4\,t \sin \pi t$
    - $x = \cos(s)$, $y = \sin(s)$
    - intersect at $t=1/4$, $s=\pi/4$

The angle looks less than $\frac{\pi}{2}$. About 1 rad?

$$\vec{r}_1'\left(\tfrac{1}{4}\right) = \begin{bmatrix} 2\sqrt{2} - \pi/\sqrt{2} \\ 2\sqrt{2} + \pi/\sqrt{2} \end{bmatrix} \qquad \left|r_1'\left(\tfrac{1}{4}\right)\right| = \sqrt{16+\pi^2}$$

$$r_2'\left(\tfrac{\pi}{4}\right) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \left|r_2'\left(\tfrac{\pi}{4}\right)\right| = 1$$

$$\vec{r}_1'\left(\tfrac{1}{4}\right)\cdot r_2'\left(\tfrac{\pi}{4}\right) = \pi = \sqrt{16+\pi^2}\cdot 1 \cdot \cos\theta$$

Therefore $\cos\theta = \dfrac{\pi}{\sqrt{16+\pi^2}}$, $\theta \doteq 0.905$ rad

Clicker quiz

What is the sine of the angle at which the helix

$\mathbf{r}(t) = (\cos(2\pi t), -\sin(2\pi t), \pi t)$

passes through the x-y plane? Hint: how is that related to the **k** component of something?

A $1/2$

B $3^{1/2}/2$

C $1/5^{1/2}$ √

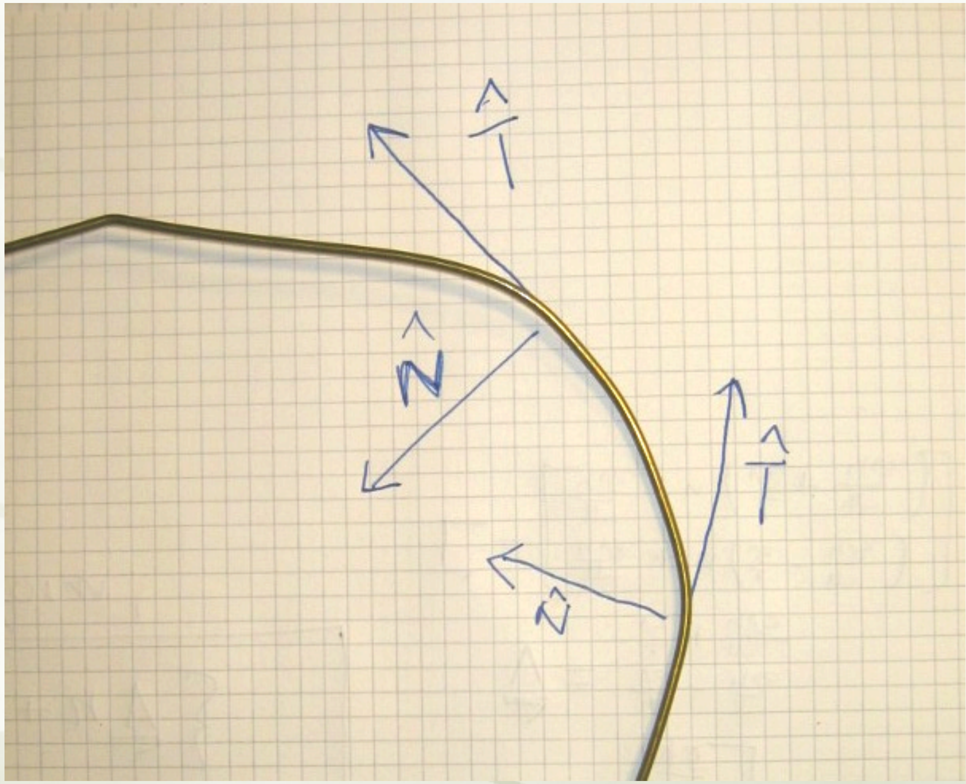
D 1

E $(2/5)^{1/2}$

F none of the above

Curves

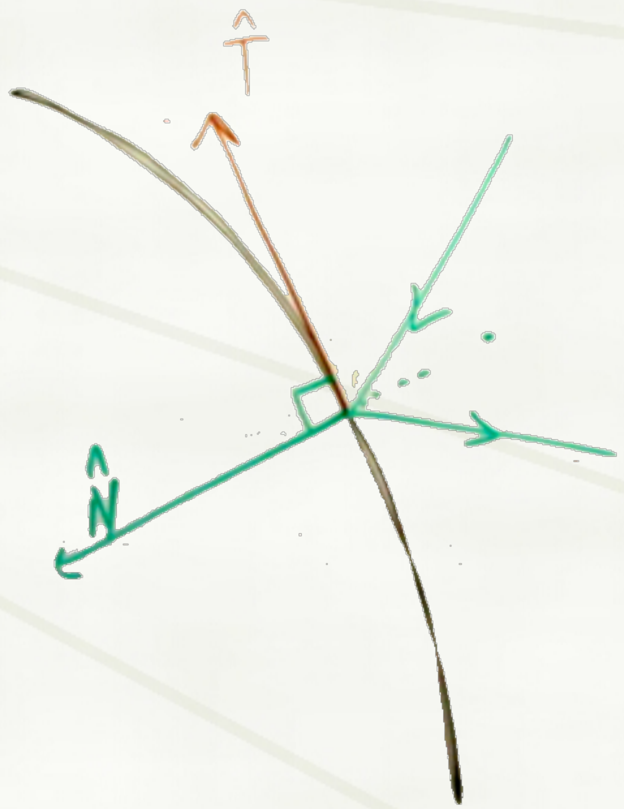
- Even if you are twisted, you have a normal!



Normal vectors

- A normal vector points in the direction the curve is bending.
- It is always perpendicular to **T**.
- What’s the formula?……………

Normal vectors and the law of reflection



Normal vectors

A measure of *curvature* κ.

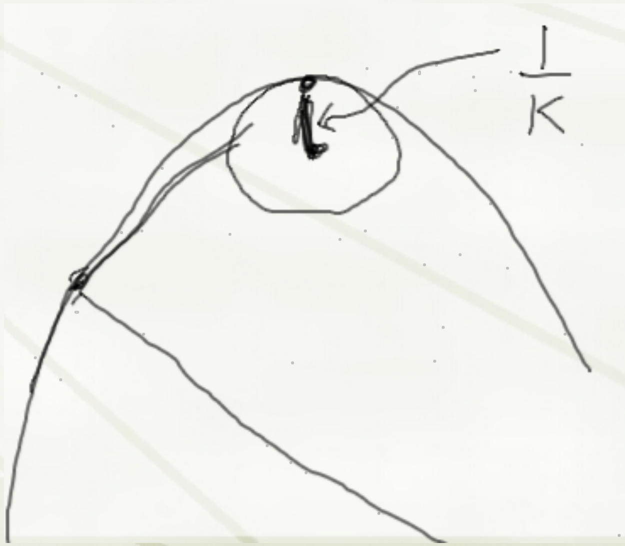
- $$\mathbf{N} = \mathbf{T}' / \|\mathbf{T}'\|.$$
- Unless the curve is straight at position P, by this definition **N** is a unit vector perpendicular to **T**. *Why?*

Normal vectors

- Another formula is $d\mathbf{T}/ds = \kappa\,\mathbf{N}$, where $\kappa = |d\mathbf{T}/ds|$ is known as *the curvature*.

Normal vectors

The radius of curvature $1/\kappa$ is the radius of the circle that best matches a curve at the point of contact.



Example: spiral in 3D

c(t) = (t Cos t, t Sin t, 0)

```
Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2 + D[r[[3]], t]^2]

Tang[r_] := {D[r[[1]], t], D[r[[2]], t], D[r[[3]], t]} / Speed[r]

General::spell : Possible spelling error: new symbol name "Tang" is similar to existing symbols {Tan, Tanh}. More…

Tang[Spiral3D[t]]
```

$$\left\{\frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2}},\ \frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2}},\ 0\right\}$$

```
Simplify[%]
```

$$\left\{\frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

The normal vector is then given by the derivative of this, scaled to length 1:

$$\mathrm{D}\left[\left\{\frac{\mathbf{Cos[t]-t\,Sin[t]}}{\sqrt{\mathbf{1+t^2}}},\ \frac{\mathbf{t\,Cos[t]+Sin[t]}}{\sqrt{\mathbf{1+t^2}}},\ \mathbf{0}\right\},\ \mathbf{t}\right]$$

$$\left\{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

```
Len[v_] := Sqrt[v[[1]]^2 + v[[2]]^2 + v[[3]]^2]
```

$$\mathbf{Len}\left[\left\{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}\right]$$

$$\sqrt{\left(\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}}\right)^2+\left(-\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}\right)^2}$$

Example: spiral in 3D

$$\sqrt{\left(\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}}\right)^2+\left(-\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}\right)^2}$$

```
Simplify[%]
```

$$\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}$$

$$\mathbf{NormVec[t]} = \left\{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}},\ -\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}\Big/\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}$$

$$\left\{\frac{\frac{-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}-\frac{t\,(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])}{(1+t^2)^{3/2}}}{\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}},\ \frac{-\frac{t\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{(1+t^2)^{3/2}}+\frac{2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}}}{\sqrt{\frac{(2+t^2)^2}{(1+t^2)^2}}},\ 0\right\}$$

Thus the normal vector reduces to

```
Simplify[%, Assumptions → {t > 0}]
```

$$\left\{-\frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

Example: spiral in 3D

Interpretation. Notice that the normal vector

$$\left\{-\frac{t\,\mathrm{Cos}[t]+\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ \frac{\mathrm{Cos}[t]-t\,\mathrm{Sin}[t]}{\sqrt{1+t^2}},\ 0\right\}$$

has no vertical component. This is because the spiral lies completely in the x-y plane, so an object moving on it is not accelerated vertically.

Example: Helix

- **Example: Helix**

```
In[5]:= Tang[Helix[t]]
```

$$\text{Out[5]=}\ \left\{-\frac{4\pi \operatorname{Sin}[4\pi t]}{\sqrt{1+16\pi^2 \operatorname{Cos}[4\pi t]^2+16\pi^2 \operatorname{Sin}[4\pi t]^2}},\ \frac{4\pi \operatorname{Cos}[4\pi t]}{\sqrt{1+16\pi^2 \operatorname{Cos}[4\pi t]^2+16\pi^2 \operatorname{Sin}[4\pi t]^2}},\ \frac{1}{\sqrt{1+16\pi^2 \operatorname{Cos}[4\pi t]^2+16\pi^2 \operatorname{Sin}[4\pi t]^2}}\right\}$$

```
In[6]:= Simplify[%]
```

$$\text{Out[6]=}\ \left\{-\frac{4\pi \operatorname{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{4\pi \operatorname{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{1}{\sqrt{1+16\pi^2}}\right\}$$

The normal vector is then given by the derivative of this, scaled to length 1:

$$\text{In[7]:=}\ \mathbf{D}\left[\left\{-\frac{4\pi \operatorname{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{4\pi \operatorname{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ \frac{1}{\sqrt{1+16\pi^2}}\right\},\ t\right]$$

$$\text{Out[7]=}\ \left\{-\frac{16\pi^2 \operatorname{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ -\frac{16\pi^2 \operatorname{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ 0\right\}$$

```
In[8]:= Len[v_] := Sqrt[v[[1]]^2 + v[[2]]^2 + v[[3]]^2]
```

$$\text{In[9]:=}\ \mathbf{Len}\left[\left\{-\frac{16\pi^2 \operatorname{Cos}[4\pi t]}{\sqrt{1+16\pi^2}},\ -\frac{16\pi^2 \operatorname{Sin}[4\pi t]}{\sqrt{1+16\pi^2}},\ 0\right\}\right]$$

$$\text{Out[9]=}\ \sqrt{\frac{256\pi^4 \operatorname{Cos}[4\pi t]^2}{1+16\pi^2}+\frac{256\pi^4 \operatorname{Sin}[4\pi t]^2}{1+16\pi^2}}$$

Example: Helix

In[9]:= $\text{Len}\left[\left\{-\frac{16\pi^2 \text{Cos}[4\pi t]}{\sqrt{1+16\pi^2}}, -\frac{16\pi^2 \text{Sin}[4\pi t]}{\sqrt{1+16\pi^2}}, 0\right\}\right]$

Out[9]= $\sqrt{\frac{256\pi^4 \text{Cos}[4\pi t]^2}{1+16\pi^2} + \frac{256\pi^4 \text{Sin}[4\pi t]^2}{1+16\pi^2}}$

In[10]:= Simplify[%]

Out[10]= $\frac{16\pi^2}{\sqrt{1+16\pi^2}}$

Remarkably, it does not depend on time. This sort of simplification is often encountered when a curve is simple or has symmetries.

In[11]:= $\text{NormVec}[t] = \text{Simplify}\left[\left\{-\frac{16\pi^2 \text{Cos}[4\pi t]}{\sqrt{1+16\pi^2}}, -\frac{16\pi^2 \text{Sin}[4\pi t]}{\sqrt{1+16\pi^2}}, 0\right\} \Big/ \left(\frac{16\pi^2}{\sqrt{1+16\pi^2}}\right)\right]$

Out[11]= {-Cos[4 π t], -Sin[4 π t], 0}

Thus the normal vector to the 3D spiral traces out a circle in 2D

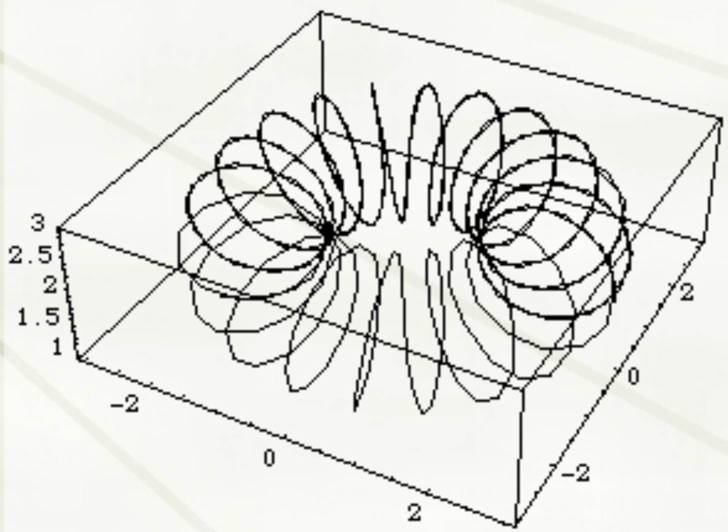
Example: Helix

Interpretation. Notice that the normal vector again has no vertical component. If a particle rises in a standard helical path, it does not accelerate upwards of downwards. The acceleration points inwards in the x-y plane. It points towards the central axis of the helix.

Example: solenoid

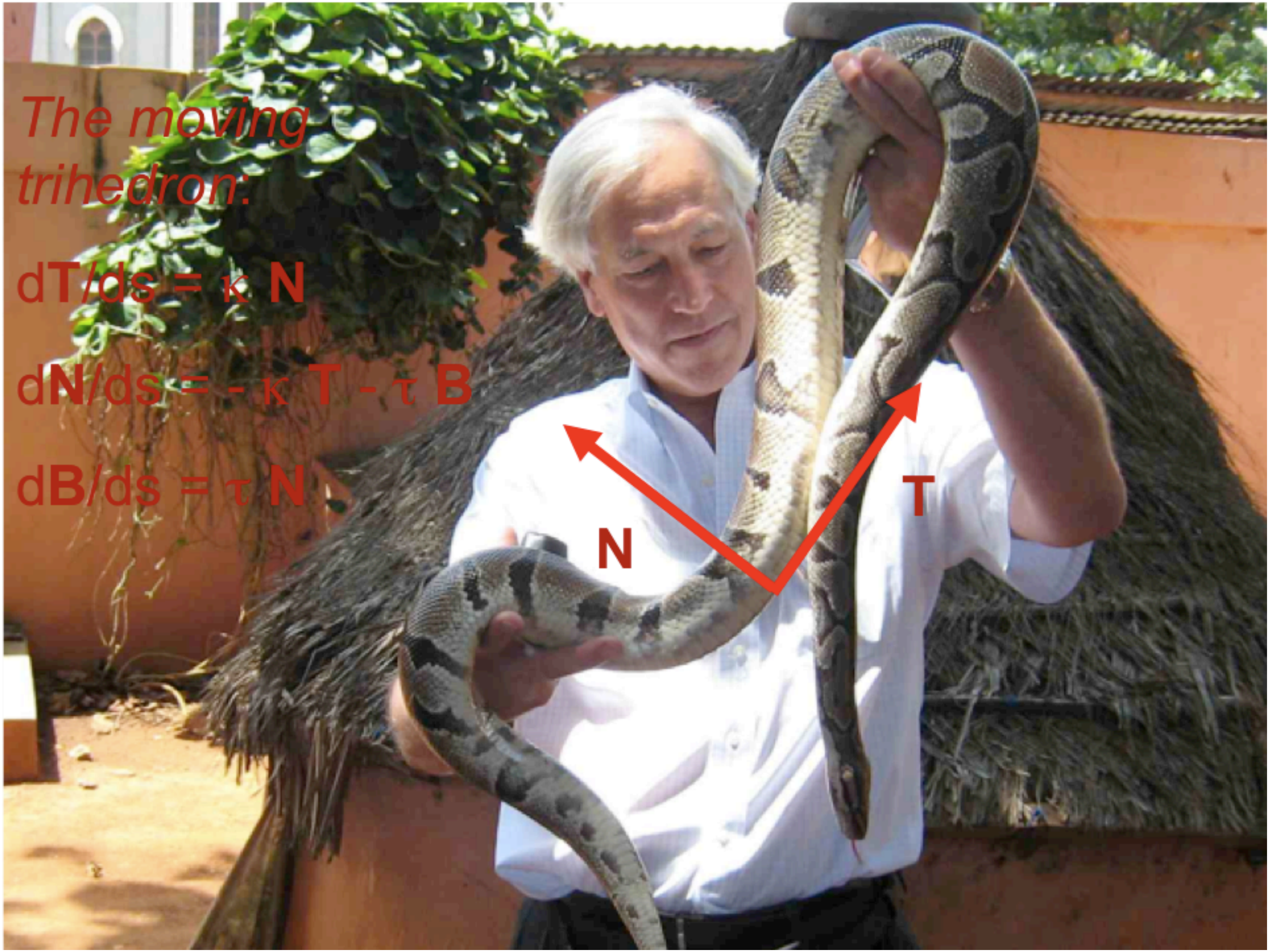
```
Solenoid[t_, r_, R_, w_] := {(R + r Cos[w t]) Cos[t], (R + r Cos[w t]) Sin[t], R + r Sin[w t]}

ParametricPlot3D[Solenoid[t, 1, 2, 20], {t, 0, 10}, PlotPoints → 360]
```



Example: solenoid

Scary, but it might be fun to work it out!


The moving trihedron:
$d\mathbf{T}/ds = \kappa\, \mathbf{N}$
$d\mathbf{N}/ds = -\kappa\, \mathbf{T} - \tau\, \mathbf{B}$
$d\mathbf{B}/ds = \tau\, \mathbf{N}$
N
T

Admittedly….

You can really get tangled up in these calculations!

Tangent and normal lines:

Ways to describe a line:

slope-intercept $y = m x + b$

2 points, point-slope

These are not so useful in 3-D.

Better:

parametric form: $\mathbf{r}(t) = \mathbf{r}_0 + u\,\mathbf{v}$ (call parameter something other than t)

Tangent and normal lines:

The essential facts about the helix:

$$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$$

$$\mathbf{T}(t) = (1/(1+16\pi^2)^{1/2})\,(-4\pi \sin(4\pi t)\,\mathbf{i} + 4\pi \cos(4\pi t)\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = -\cos(4\pi t)\,\mathbf{i} - \sin(4\pi t)\,\mathbf{j}$$

Example: Tangent and normal lines at (1,0,1)

Tangent and normal lines:

Example: Tangent and normal lines at (1,0,1)

$\mathbf{r}(t) = (\cos(4\pi t), \sin(4\pi t), t)$

$= (1,0,1)$ when $t = 1$.

$\mathbf{T}(1) = (1/(1+16\pi^2)^{1/2})\ (-4\pi \sin(4\pi)\ \mathbf{i} + 4\pi \cos(4\pi)\ \mathbf{j} + \mathbf{k})$

$= (1/(1+16\pi^2)^{1/2})\ (4\pi\ \mathbf{j} + \mathbf{k})$.

Line: $(1,0,1) + u\ (4\pi\ \mathbf{j} + \mathbf{k})$

Hey! What in &#$ happened to the* $(1/(1+16\pi^2)^{1/2})$?

Tangent and normal lines:

Example: Tangent and normal lines at (1,0,1)

$\mathbf{r}(t) = (\cos(4\pi t), \sin(4\pi t), t)$

$= (1,0,1)$ when $t = 1$.

$\mathbf{N}(1) = -\cos(4\pi)\,\mathbf{i} - \sin(4\pi)\,\mathbf{j}$

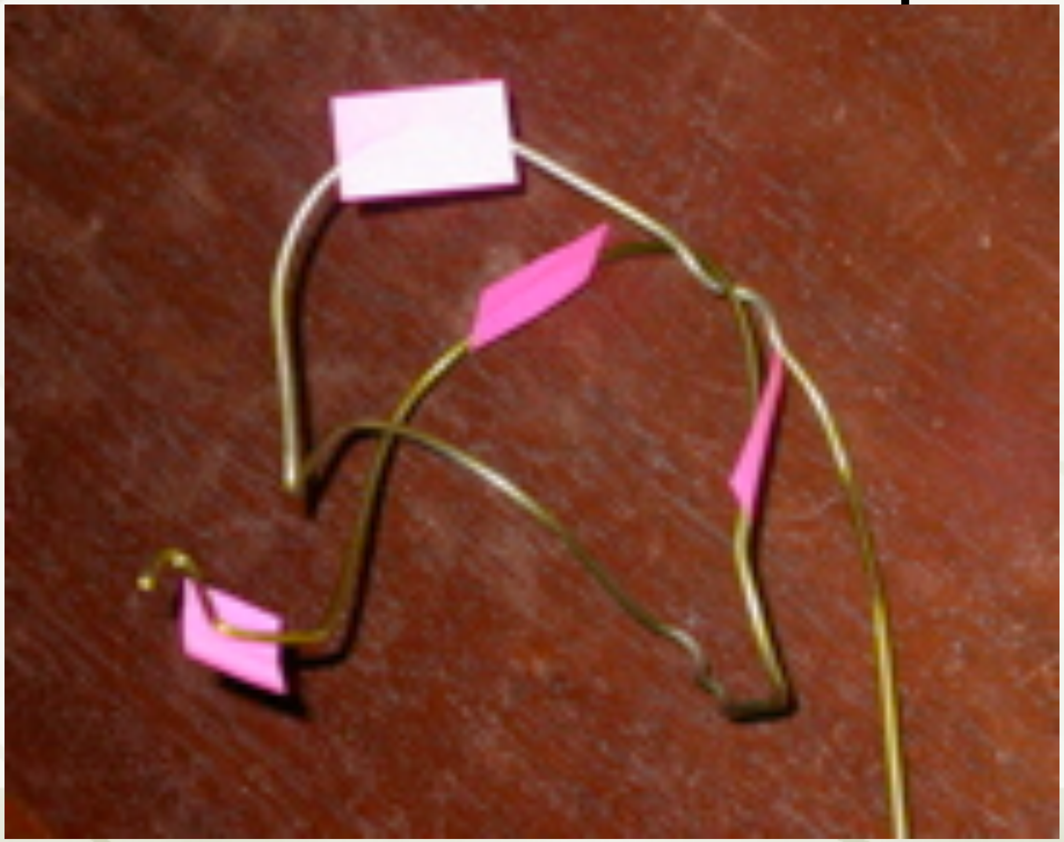
$= -\,\mathbf{i}$.

Line: $(1,0,1) + u\,\mathbf{i}$.

Wait a minute! What about the sign ?

The osculating plane

- Bits of curve have a “best plane.”



stickies on wire.

Each stickie contains **T** *and* **N**.

The osculating plane

- Bits of curve have a “best plane.”

- One exception - a straight line lies in infinitely many planes.

The osculating plane

- What's the formula, for example for the helix?

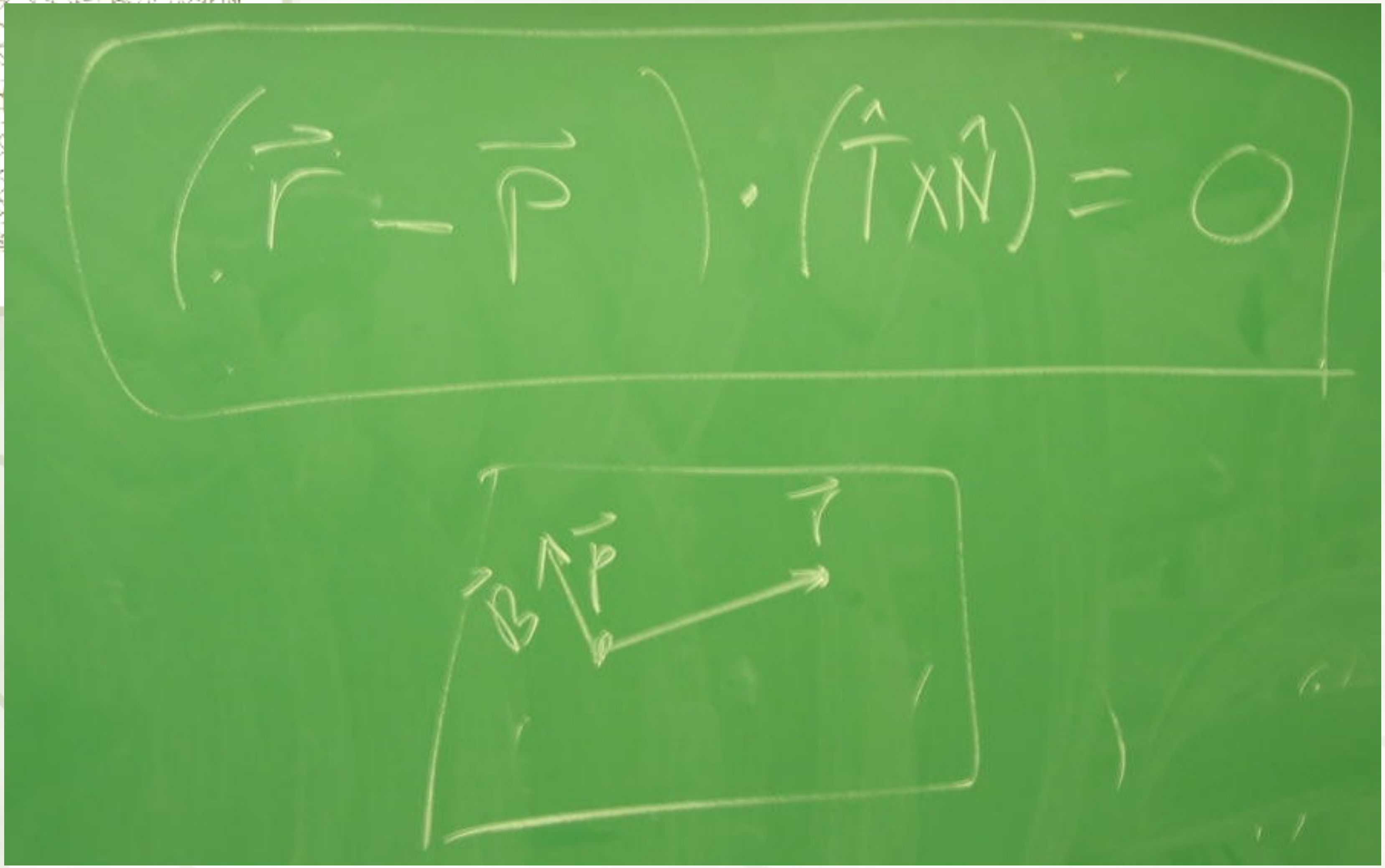
1. Parametric form

2. Single equation

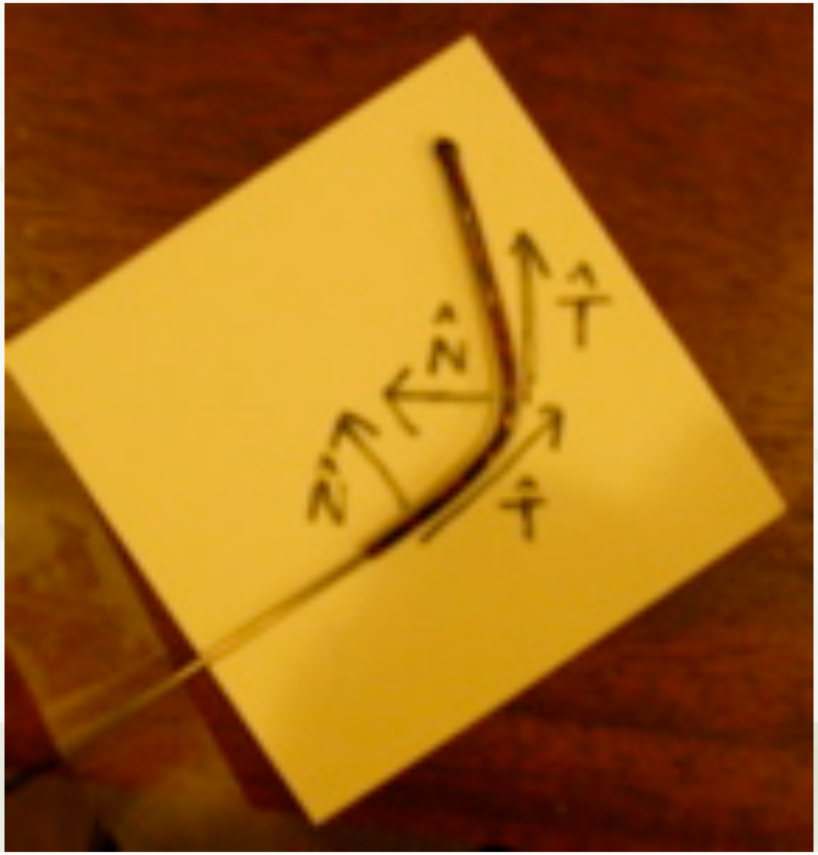
The binormal $\mathbf{B}$

- The normal vector to a plane is *not* the same as the normal to a curve in the plane. It has to be $\perp$ to all the curves and vectors that lie within the plane.
- Since the osculating plane contains $\mathbf{T}$ and $\mathbf{N}$, a normal to the plane is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$


$(\vec{r}-\vec{p})\cdot(\hat{T}\times\hat{N})=0$
$\vec{B}$
$\vec{p}$
$\vec{r}$

Close-up

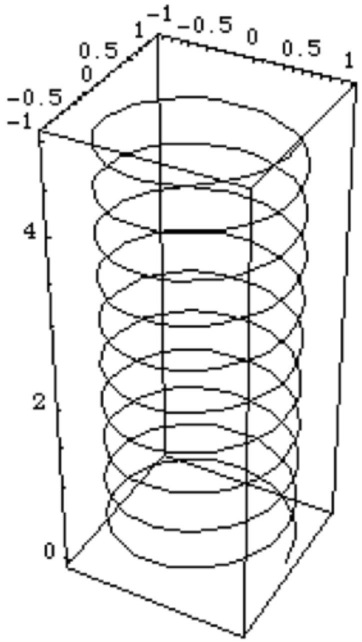


The osculating plane

- What's the formula, for example for the helix?

1. Parametric form

2. Single equation



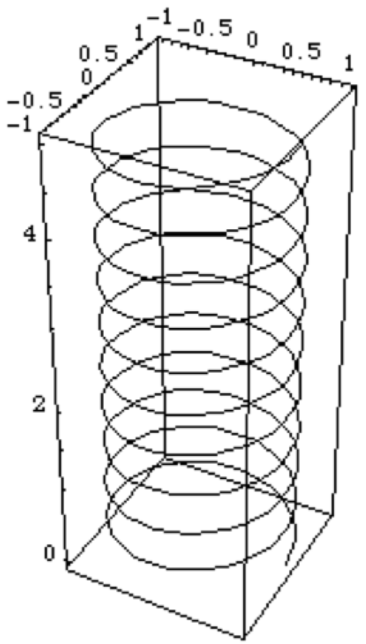
Example: The helix

r(t) = cos(4 π t) **i** + sin(4 π t) **j** + t **k**

T(t) = (- 4 π sin(4 π t) **i** + 4 π cos(4 π t) **j** + **k**)/$(1+16\pi^2)^{1/2}$

N(t) = - cos(4 π t) **i** - sin(4 π t) **j**

Example: The helix



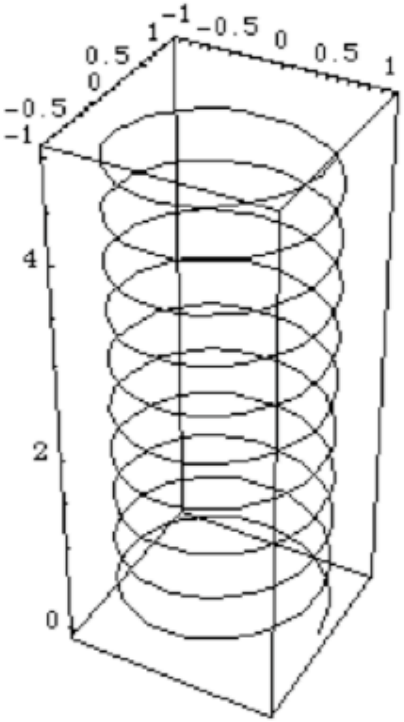
$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$

$\mathbf{T}(t) \times \mathbf{N}(t) = (-\sin(4\pi t)\,\mathbf{i} + \cos(4\pi t)\,\mathbf{j} - 4\pi\,\mathbf{k})/(1+16\pi^2)^{1/2}$

Osculating plane at (1,0,1): Calculate at t=1.

$(\mathbf{r}_{osc} - (\mathbf{i} + \mathbf{k})) \cdot (1\,\mathbf{j} - 4\pi\,\mathbf{k}) = 0$

(The factor $(1+16\pi^2)^{1/2}$ can be dropped.)



Example: The helix

In coordinates,

$$x_{helix}(t) = \cos(4\ \pi\ t),$$

$$y_{helix}(t) = \sin(4\ \pi\ t)$$

$$z_{helix}(t) = t$$

And

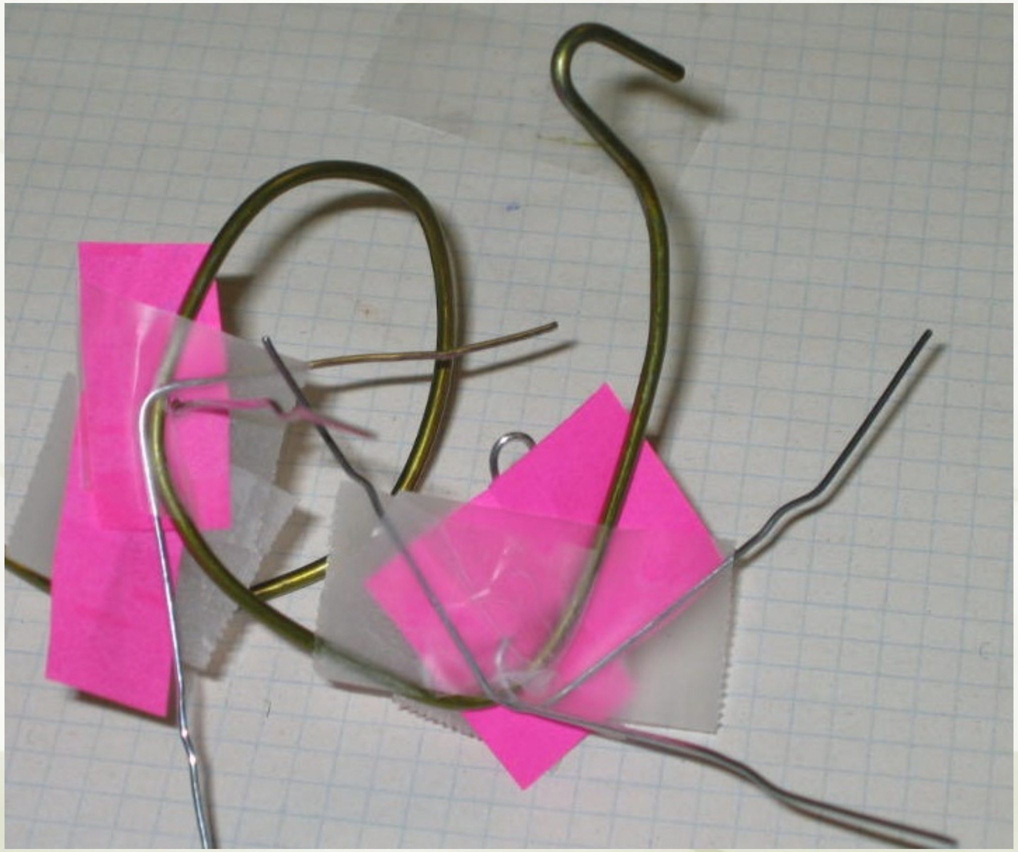
$$(x_{osc} - 1)\cdot 0 + (y_{osc} - 0)\cdot 1 + (z_{osc} - 1)\cdot(-4\pi) = 0,$$

Which simplifies to:

$$\mathbf{y_{osc} - 4\ \pi\ z_{osc} = - 4\ \pi}$$

The moving trihedron

- The curve's preferred coordinate system is oriented along (**T**,**N**,**B**), *not* some Cartesian system (**i**,**j**,**k**) in the sky.



The moving trihedron

- A vehicle can rotate around any of these axes. A rotation around **T** is known as *roll*. If the vehicle has wings (or a hull) it may prefer a second direction over **N**. For example, the wing direction may correlate with **N** when the airplane turns without raising or lowering the nose. Such an acceleration is called *yaw*.

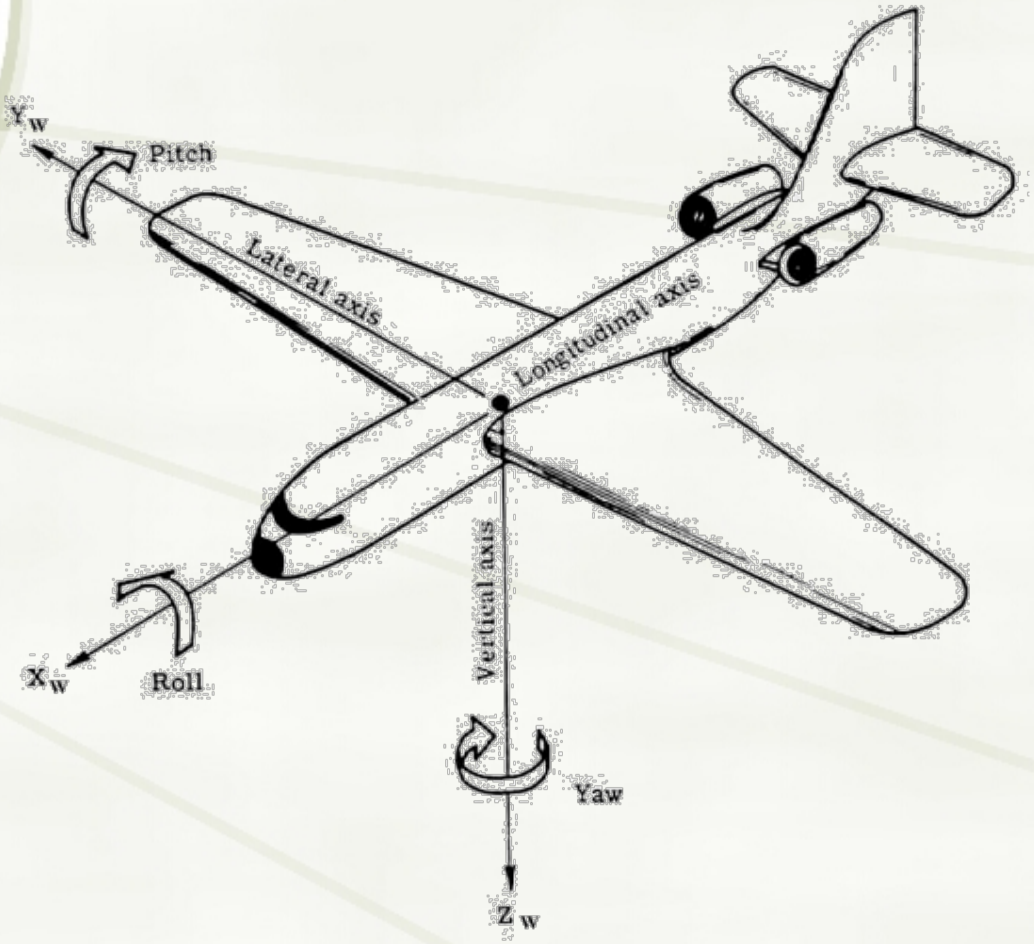
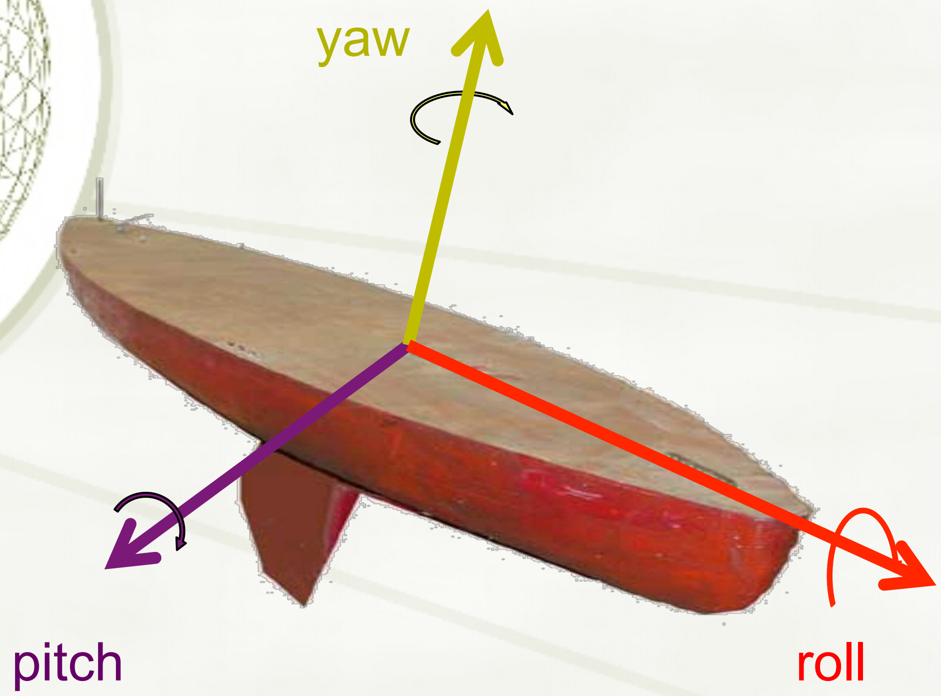


Figure from JPL/NASA

The moving trihedron

- However, when the aircraft soars or dives (this kind of acceleration is called *pitch*), the normal vector **N** is perpendicular to the wing axis, which in this case correlates with the binormal **B**.

- An aircraft can accelerate, roll, yaw, and pitch all at once. *Fasten your seatbelt!*



Watercraft have the same kinds of accelerations as aircraft. The rudder controls yaw. The boat is usually designed to minimize pitch and roll.