MATH 2411 - Harrell

Which way is up?

Lecture 8

Clicker quiz

What is $\frac{\partial F}{\partial x}(\pi, 1)$ when $F(x,y) = \frac{x}{y}\sin\left(\frac{2x}{y}\right)$?

A 0

B 2 π i

C 2 π √

D none of the above.

A frank talk about sets

- Neighborhood of a point
- Interior, boundary
- Open
- Closed
- Neither open nor closed

A set does not necessarily have boundaries

If it does have boundaries…

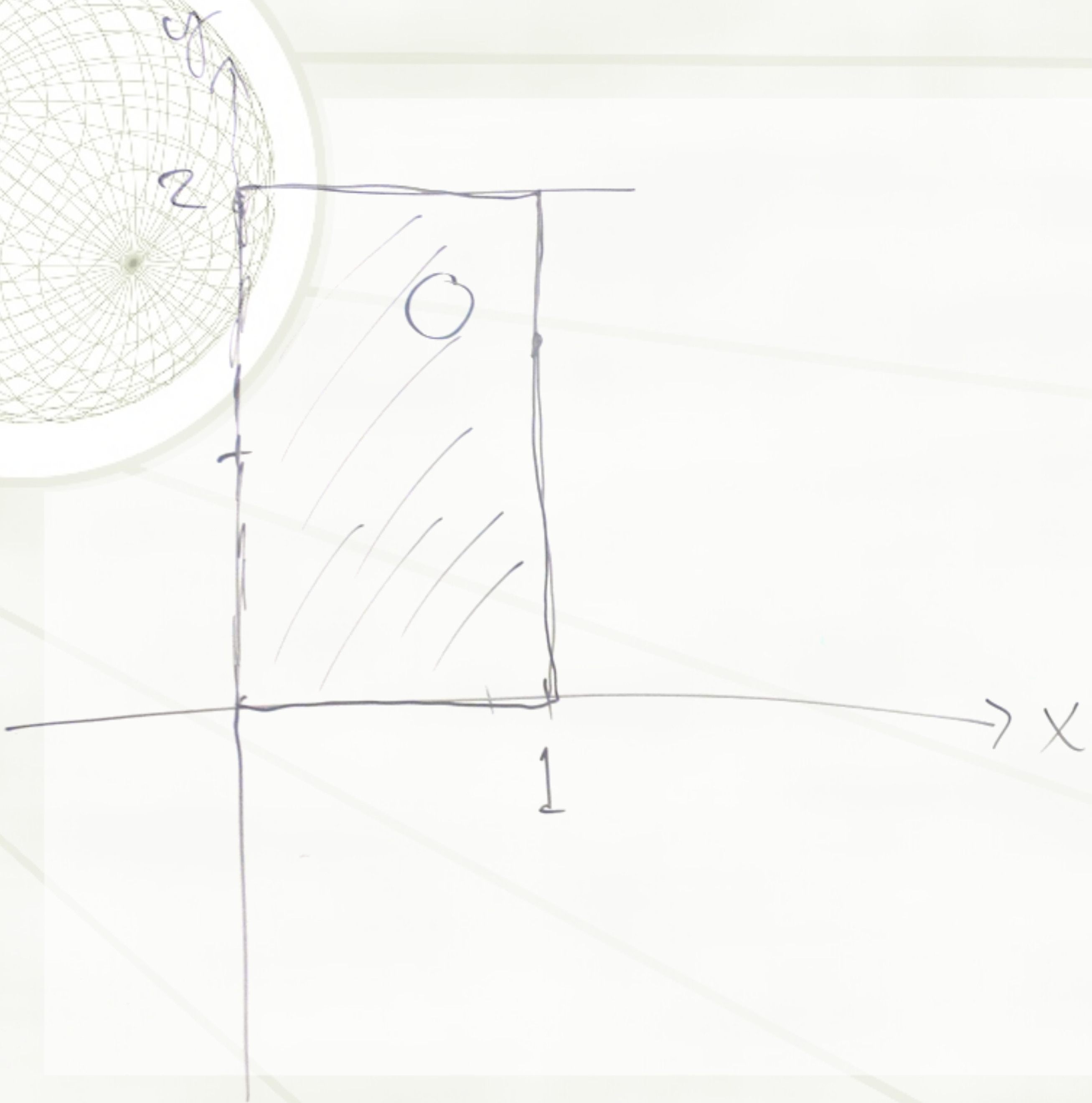
Are the boundaries part of the set or not?

A frank talk about sets

- Neighborhood of a point
- Interior, boundary
- Open – *every point is interior*
- Closed - *contains all of its boundary*
- Neither open nor closed

Open, closed, or neither?

- Closed rectangle:
  - $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$
- Open rectangle:
  - $\{(x,y) : 0 < x < 1,\ 0 < y < 2\}$
- Neither one:
  - $\{(x,y) : 0 \le x < 1,\ 0 \le y \le 2\}$


y
2
D
1
x

Partial derivatives as a vector

- In 2 dimensions, we have 2 partials,

  $\partial F/\partial x$ and $\partial F/\partial y$

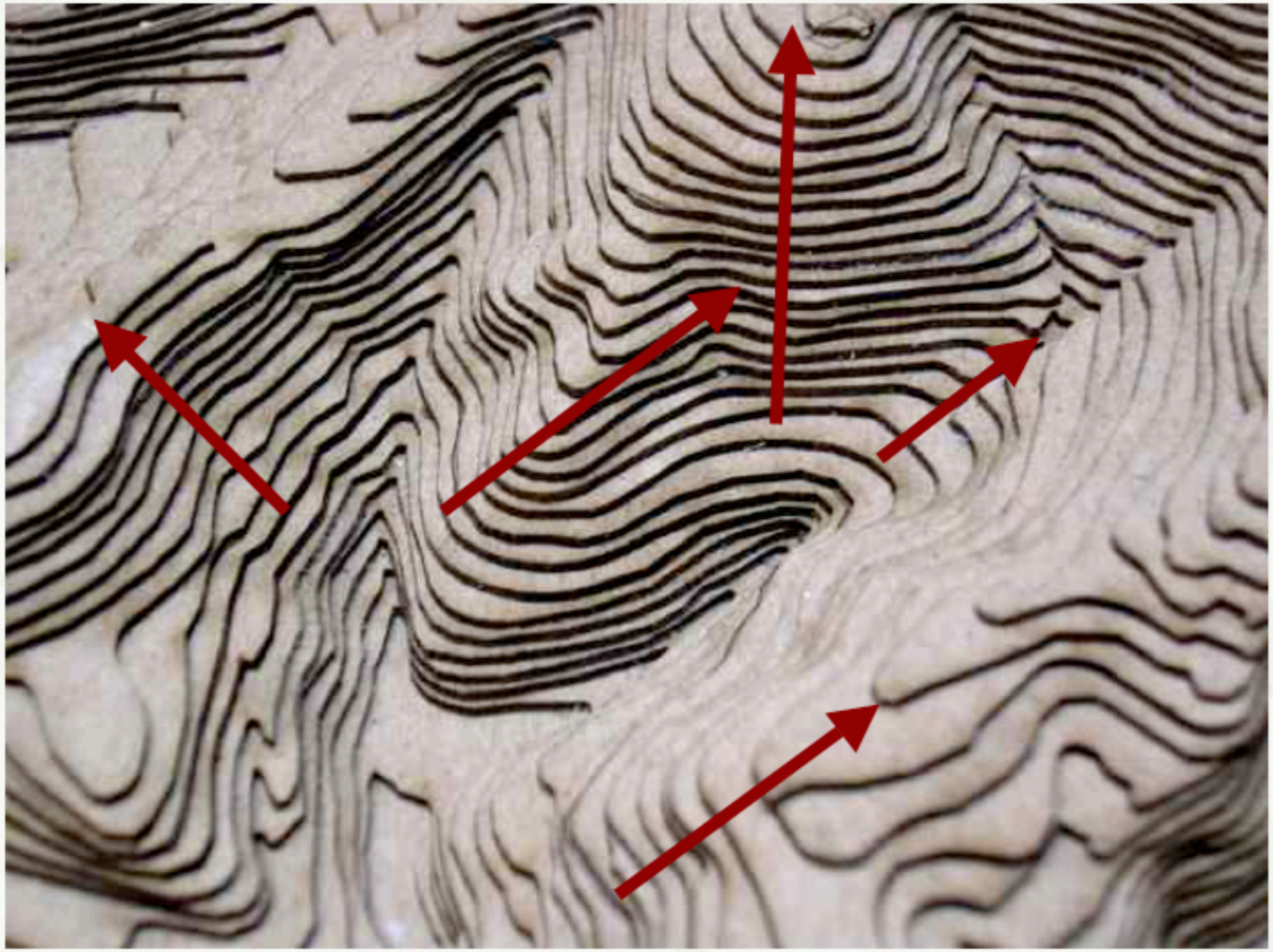
- In 3 dimensions we have 3

  $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$

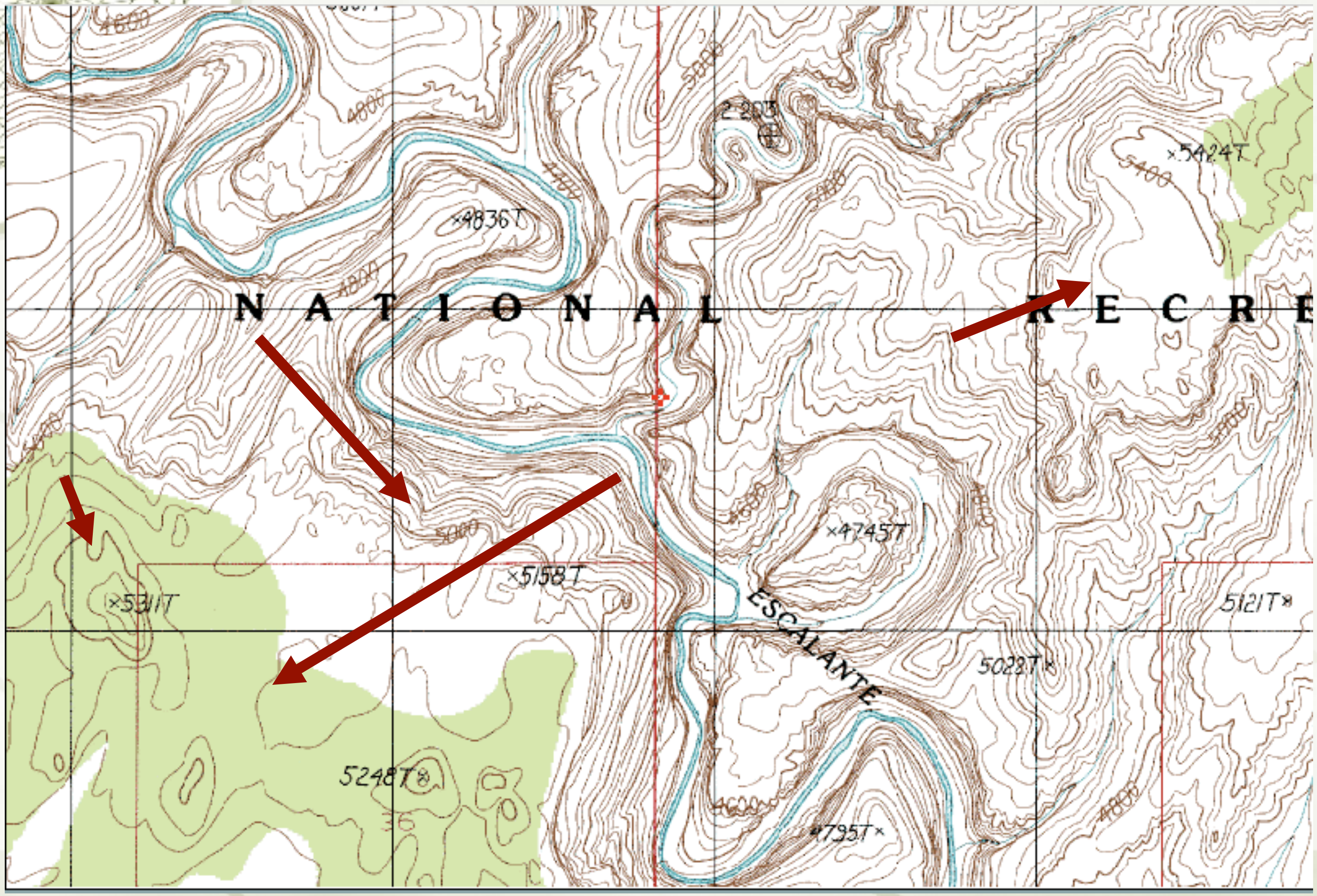
- Same number as components of a vector, *hmmmmm*.

Point uphill

Our convention is that the slope vector applies to the point at its base, not its tip.



Point uphill



http://commons.wikimedia.org/wiki/Image:Ut_escalante_canyons1.jpg, with permission

How to determine the gradient from a topographic map

- At a point on the Escalante map, we can move uphill by _3_ contours, separated by 40 feet, over a distance _1/7_ mile,
  - rise = _3_*40 = _120_
  - run = _700ft_
  - slope = ________ 12/70

"Uphill" as a vector

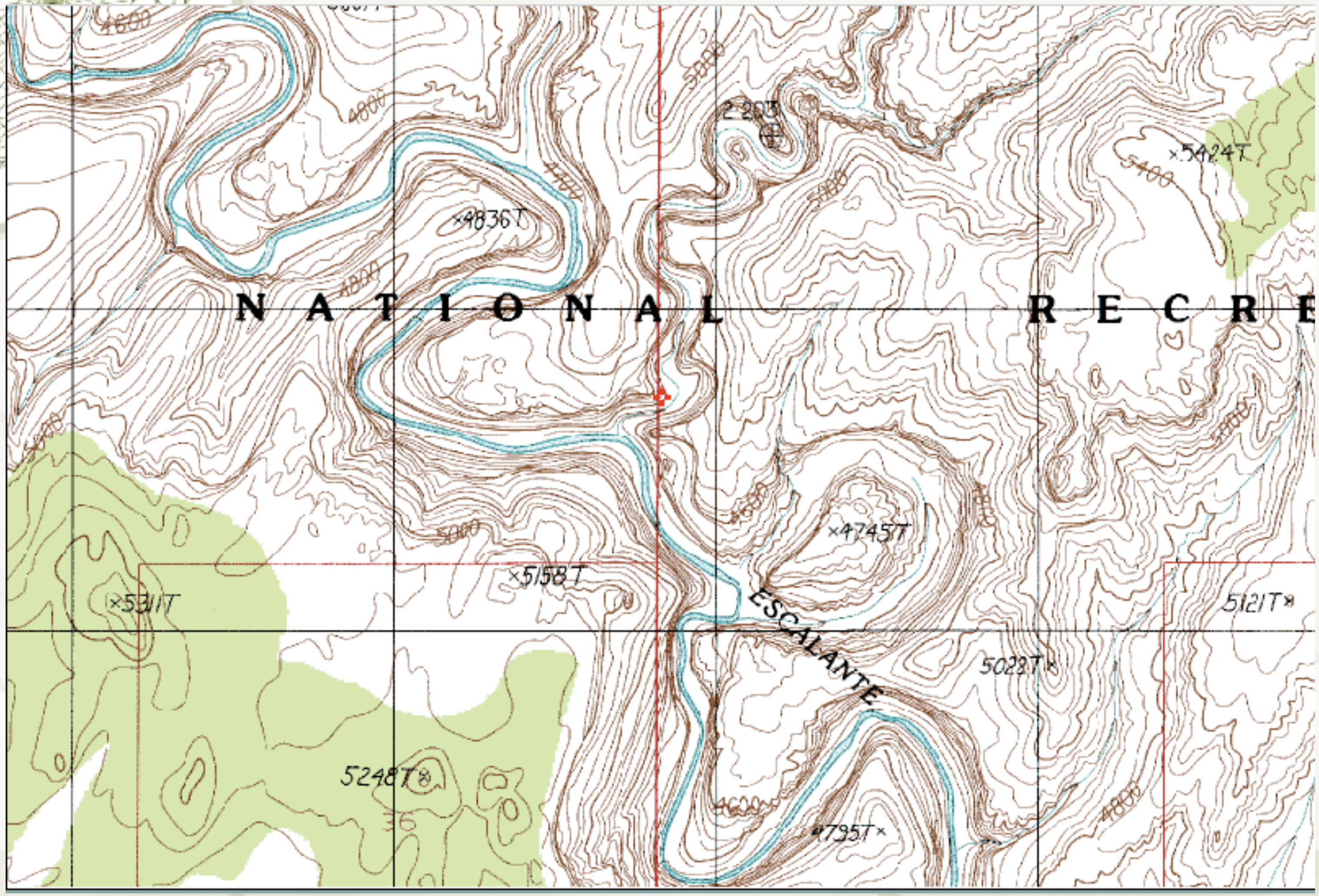
- Direction
- Magnitude = the slope in that direction, e.g., 5% grade.
- If it happens that "uphill" is along the positive x-axis, then the slope is

$$\partial f/\partial x.$$

Max/Min problems

- At a local max or min, which way is up?

Point uphill

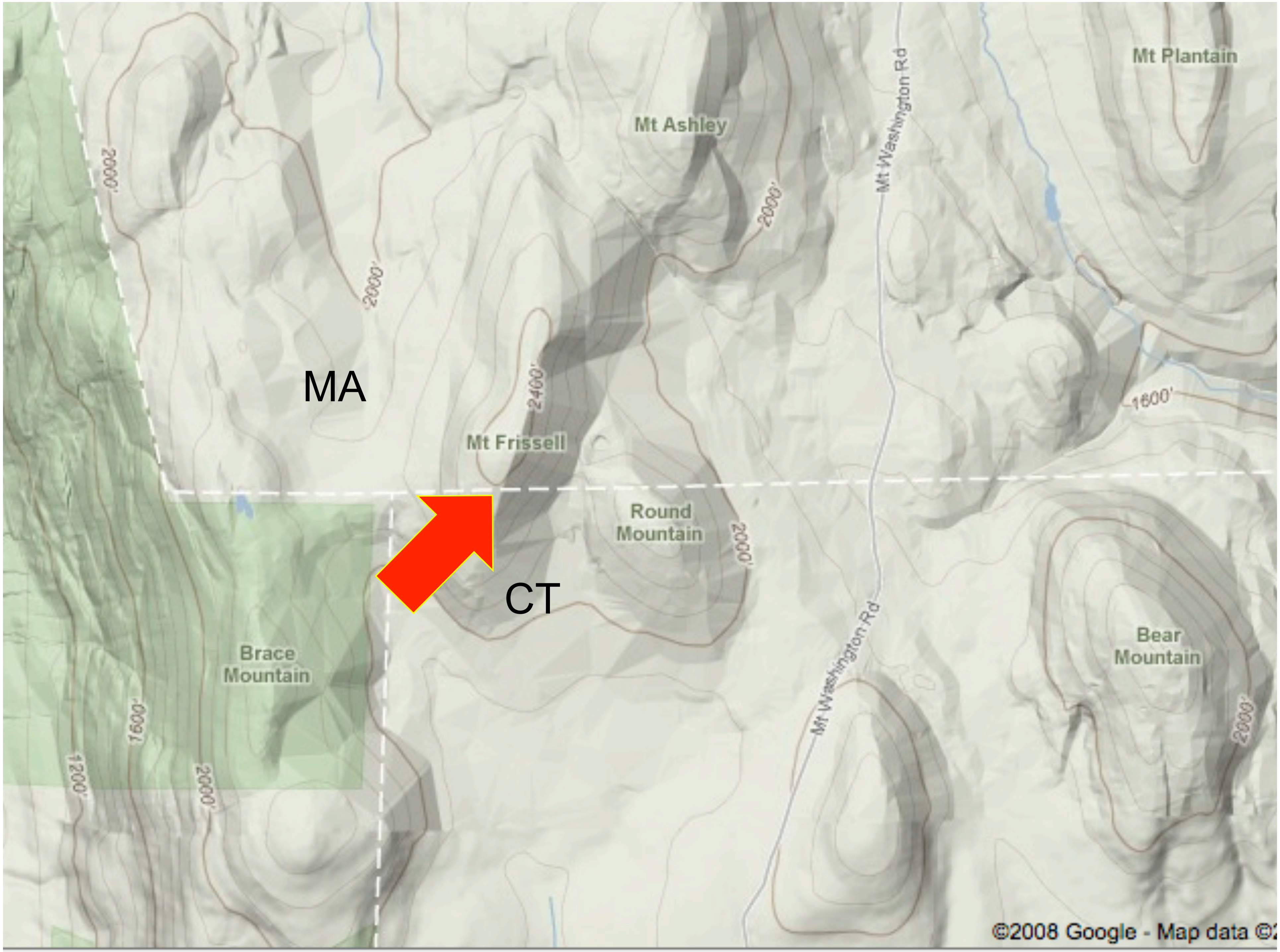


An important theorem

- A continuous function on a closed and bounded set *always* has a maximum value and a minimum value.
- Might or might not be true if the set is unbounded or not closed.

An important theorem

- A continuous function on a closed and bounded set *always* has a maximum value and a minimum value.
- Yet the gradient need *not* be **0** at the maximum.



Mt Plantain
Mt Ashley
Mt Washington Rd
2000'
2400'
1600'
MA
Mt Frissell
Round Mountain
CT
Brace Mountain
Bear Mountain
1200'
©2008 Google - Map data ©

Directional derivative

- Choose a direction, by taking a unit vector **u**.
- Measure rise/run over a small distance in the direction **u**.

Directional derivative

- What is the d.d. if the vector **u** is *tangent* to a level curve?
- What is the d.d. if the vector **u** is *normal (perpendicular)* to a level curve?
- Of all directions **u**, which one gives the greatest d.d.?

Differentiation as a *vectorial* concept

- Since you can't divide by a vector,
- We say f is differentiable iff

  $$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{y} \bullet \mathbf{h} + o(\mathbf{h}),$$

- $\mathbf{y}$ is called the *gradient* of f at x, and denoted $\nabla f(\mathbf{x})$.
- *The gradient* $\nabla f(\mathbf{x})$ *of a scalar function is a vector-valued function of a vector variable!*

- ∇
- “grad”
- “del”
- “nabla”

Some good news:

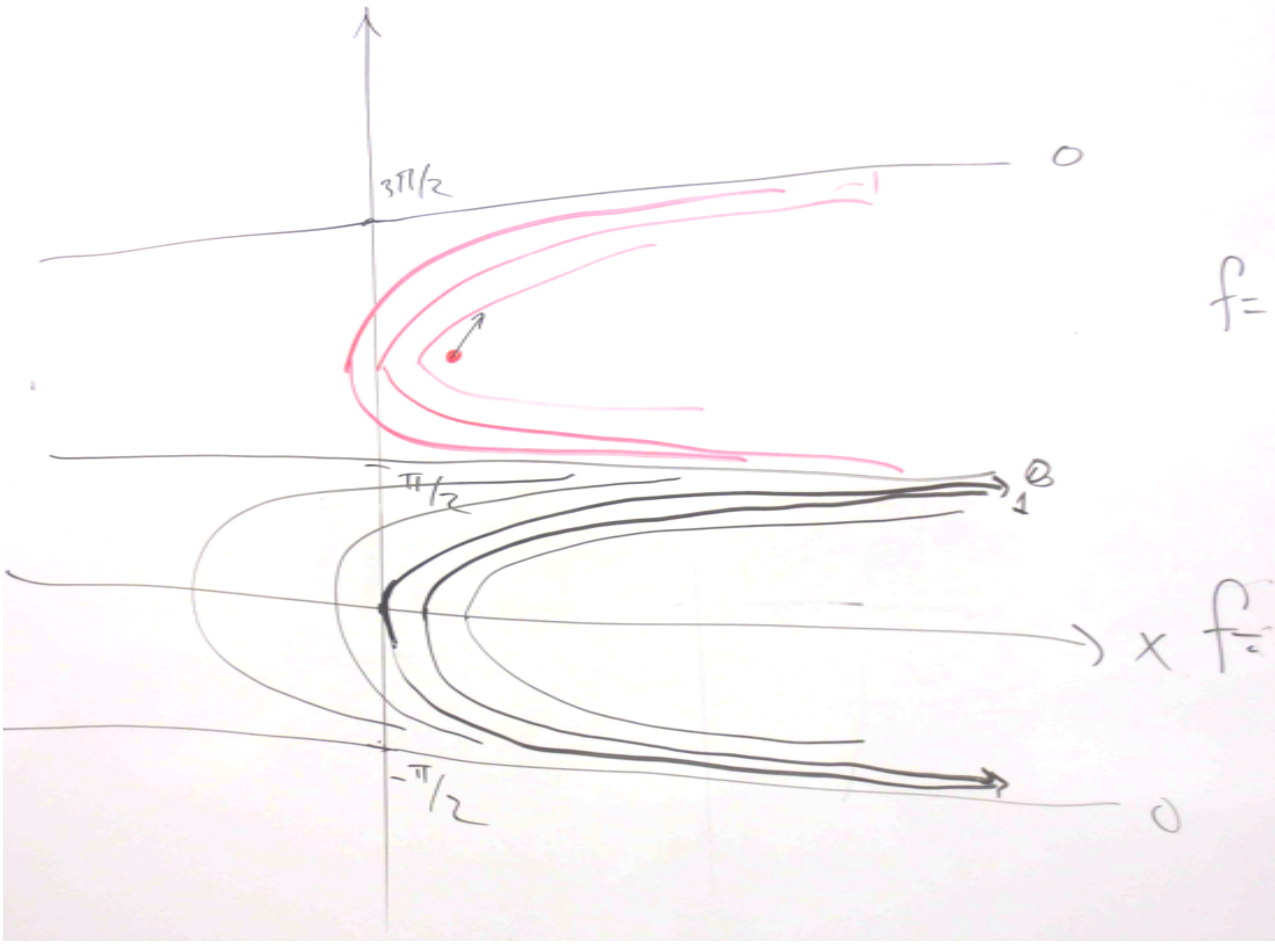
- It is easy to calculate the gradient.
- With the gradient you can easily calculate directional derivatives.
- "Uphill" is nothing other than the gradient!
- Tangent planes are also not hard to work out.

Now let's do some examples

- Let's consider $f(x,y) = e^x \cos y$. (You might see this function in a thermodynamics class, since it is a possible equilibrium temperature distribution in a homogeneous solid.)

Example: $f(x,y) = e^x \cos y$

Show gradient on contour plot (first with no calculations)



3π/2
π/2
-π/2
x
0
0
-1
1
f=
f=

Example: $f(x,y) = e^x \cos y$

$\nabla f = e^x \cos y\ \mathbf{i} - e^x \sin y\ \mathbf{j}$

Magnitude is e^x.

Example: $f(x,y) = e^x \cos y$

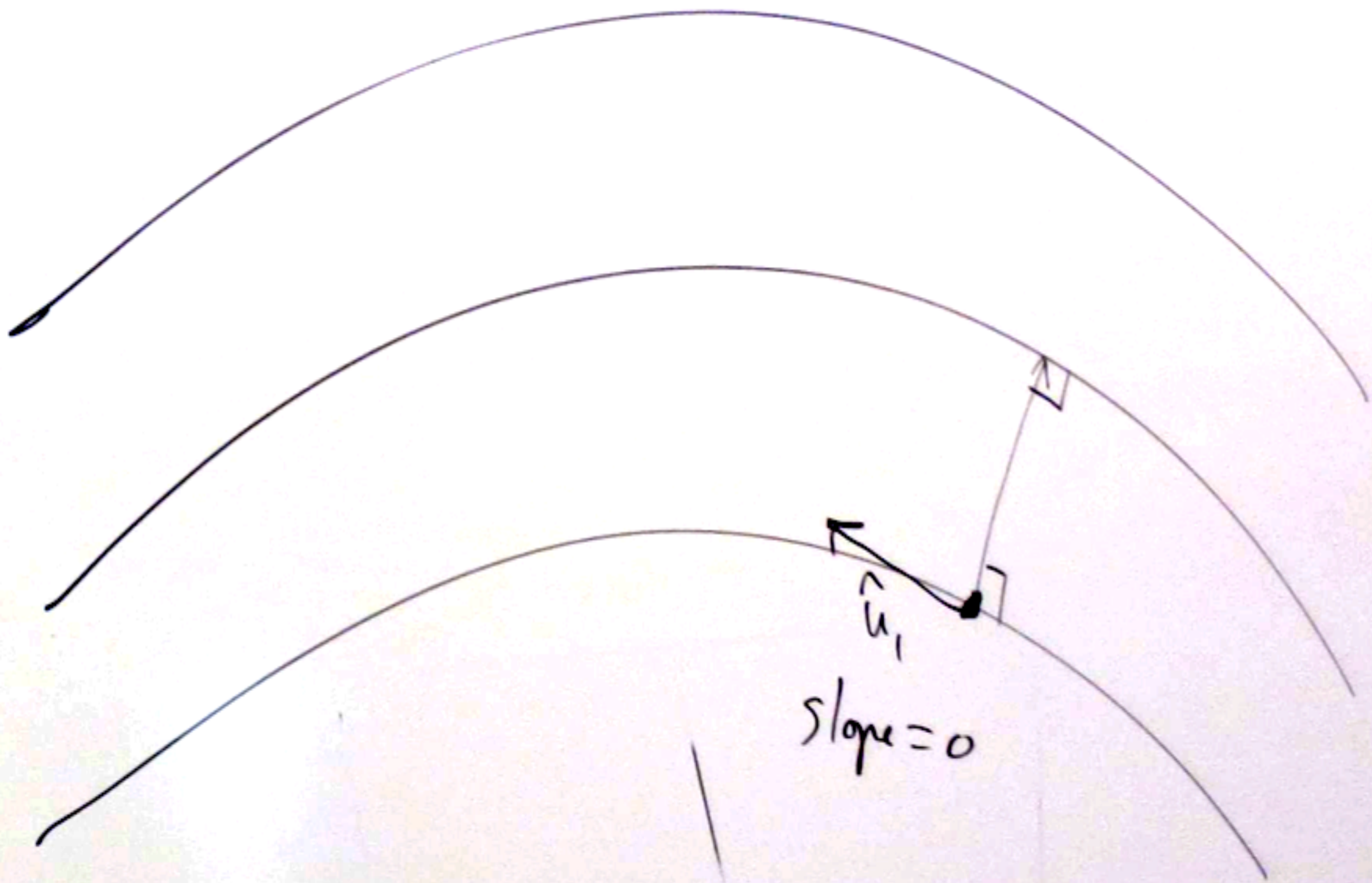
Find the directional derivative in the "northeast" direction **i** + **j**, at P = $(1,\pi)$.

1. Calculate the gradient
2. Identify a *unit* vector in the direction you want: $2^{-1/2}\,\mathbf{i} + 2^{-1/2}\,\mathbf{j}$.
3. Take dot product

ANSWER: $2^{-1/2}\, e^{1}(\cos \pi - \sin \pi) = -e/2^{1/2}$.

How to determine the gradient from a topographic map

- Direction - perpendicular to the “contour” (= level curve) passing through the position of interest.
- Calculate slope by measuring off a short distance and counting the contours crossed in that distance.
- Given a direction and a magnitude, you have the vector.



$\hat{u}_1$
slope = 0

A problem from an old final exam:

A topographic map such as the one shown here, of the Chattahoochee National Recreation Area in Sandy Springs, is a contour plot for a function f(x,y). The contours are level sets for values of z=f(x,y). Contours are separated by heights of 10 feet (every fifth contour is printed darker). The horizontal scale is such that the square shown is 2500 feet on a side.

The point P

Annotate the contour map as follows:

a) Find the top of a hill and label it with the letter **T**.

b) Find a saddle point and label it with the letter **S**.

c) There are several cliffs in the park. Find a cliff on this map and label it with the letter **C**.

d) In the lower left part of the map you will find a small arrow. Estimate the gradient at the point of the arrow. By the way, the arrow drawn is not meant to indicate the gradient at its base.) Draw a vector on the map with the same direction as the gradient, and estimate the magnitude of the gradient here:________________

Tangent planes

- Tangent plane passing through $\mathbf{r}_0$ and normal to the vector $\mathbf{N}$:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N} = 0$$

Claim

The normal to the surface

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, \pm 1\right)$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$z - \frac{\partial f}{\partial x} x - \frac{\partial f}{\partial y} y = \#$$

$$N = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

Strategies for working with the gradient

- Reduce a problem to 1-D
- Keep the problem 3-D, let 1-D calculus be your guide.

Intermediate value theorem

- If f is differentiable on the line segment **ab**, there exists a position **c** so that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \bullet (\mathbf{b} - \mathbf{a})$$

- Why is this not as great as in 1D?

Intermediate value theorem

- If f is differentiable on the line segment **ab**, there exists a position **c** so that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \bullet (\mathbf{b} - \mathbf{a})$$

- Why is this not as great as in 1D?

  A: the gradient doesn't have to point in the direction ($\mathbf{b} - \mathbf{a}$), and $|\nabla f(\mathbf{c})|$ isn't the average slope $|f(\mathbf{b}) - f(\mathbf{a})| / |\mathbf{b} - \mathbf{a}|$.

Intermediate value theorem

- If f is differentiable on the line segment **ab**, there exists a position **c** so that

  $$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \bullet (\mathbf{b} - \mathbf{a})$$

- Example: (0,0) to (1,0), f(x,y) = x+y.

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla g$ on U, then

$$f(\mathbf{x}) = g(\mathbf{x}) + C.$$

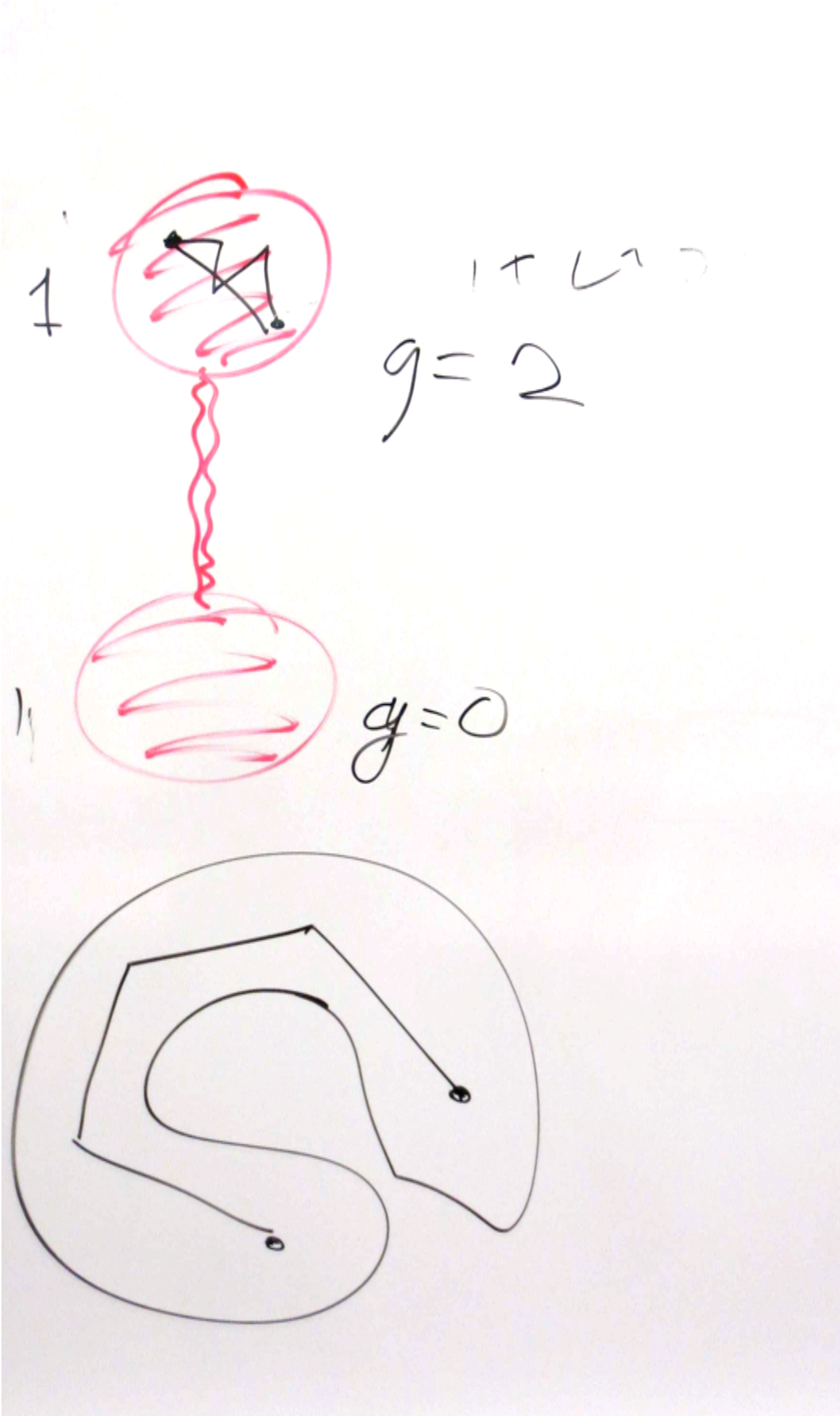
- How would you prove this?

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla g$ on U, then

  $$f(\mathbf{x}) = g(\mathbf{x}) + C.$$

- How would you prove this?
- Good enough to replace f by f-g and show that is $\nabla f = 0$ on U, then f is a constant.
- Connect any two points by “polygonal path” and use M.V. Theorem.


g=2
g=0

Gradient determines f up to a constant

- Let U be open and connected, and f and g be differentiable on U. If $\nabla f = \nabla g$ on U, then

  $$f(\mathbf{x}) = g(\mathbf{x}) + C.$$

- How about an example?

- Arctan(y/x) vs. Arccos$(x/(x^2+y^2)^{1/2})$