The space of bounded linear transformations from a Banach space $\mathcal{B}_1$ to a Banach space $\mathcal{B}_2$ is a Banach space, with the operator norm,

$$\|T\|_{op} = sup_{\|x\|_1=1}\|Tx\|_2.$$

**Proof**. To verify that $\|\cdot\|_{op}$ is a norm, we need to show
1. $\|T\|_{op} \geq 0$ and $= 0$ iff $T = 0$. Immediate from the definition of the operator norm.
2. $\|\alpha T\|_{op} = |\alpha|\|T\|_{op}$, also immediate, from the same property for $\|\cdot\|_2$, since $\|\alpha Tx\|_2 = |\alpha|\|Tx\|_2$.
3. The triangle inequality, $\|S + T\|_{op} \leq \|S\|_{op} + \|T\|_{op}$. For this,

$$\|S + T\|_{op} == sup_{\|x\|_1=1}\|(S + T)x\|_2 \leq sup_{\|x\|_1=1}\left(\|Sx\|_2 + \|Tx\|_2\right)$$

$$\leq sup_{\|x\|_1=1}\|Sx\|_2 + sup_{\|x\|_1=1}\|Tx\|_2 = \|S\|_{op} + \|T\|_{op}.$$

To verify completeness, suppose that $\{T_n\}$ is a Cauchy sequence in the operator norm. Then since $\|T_n(x) - T_m(x)\|_2 = \|\left(T_n - T_m\right)(x)\|_2 \leq \|T_n - T_m\|_{op}\|x\|_1$, it follows that $\{T_n(x)\}$ is Cauchy in $\mathcal{B}_2$. therefore it has a limit $z \in \mathcal{B}_2$, and we define $T_*$ as the operator mapping $x$ to $z$. It is mechanical to check that $T_*$ is linear. To see that $T_*$ is bounded, notice that $\|T_*(x)\|_2 \leq limsup_n\|T_n\|_{op}\|x\|_1$. Since $\{T_n\}$ is Cauchy, for any $\epsilon > 0$ there is a fixed $m$ so that for $n > m, \|T_n - T_m\|_{op} < \epsilon$, and therefore $\|T_*(x)\|_2 \leq \left(\|T_m\|_{op} + limsup_{n>m}\|T_n - T_m\|_{op}\right)\|x\|_1 \leq \left(\|T_m\|_{op} + \epsilon\right)\|x\|_1$. This shows that $T$ is a bounded operator.