

An important consequence of the Projection theorem is the

Riesz lemma. Let ϕ be a continuous linear functional on \mathcal{H} , a Hilbert space. Then there exists a unique vector $v \in \mathcal{H}$ (depending on ϕ), such that for all $x \in \mathcal{H}$,

$$\phi(x) = \langle x, v \rangle.$$

Proof. If ϕ is the 0 functional, then $v = 0$. Hence we assume ϕ is not the 0 functional. Let \mathcal{N} be the null space of ϕ . This is a closed subspace, and there must be some vector, z outside \mathcal{N} , since ϕ is not the 0 functional. By the projection lemma, z can be written uniquely as

$$z = u + w,$$

with w orthogonal to \mathcal{N} .

The claim is that the desired v is given by

$$\frac{\phi(\bar{w})}{\|w\|^2} w.$$

Here is the calculation. Since for any vector x ,

$$\phi\left(\left(x - \frac{\phi(x)}{\phi(w)} w\right)\right) = \phi(x) - \phi(x) = 0,$$

we see

$$\left(x - \frac{\phi(x)}{\phi(w)} w\right) \in \mathcal{N},$$

and for every vector u in \mathcal{N} we know $\phi(u) = \langle u, w \rangle = 0$. On the other hand, $\langle w, v \rangle = \frac{\phi(w)}{\|w\|^2} \langle w, v \rangle = \phi(w)$, which shows that the functional ϕ is the same as the inner product with v for vectors parallel to w .

The clincher is then that any vector x of \mathcal{H} , is a linear combination of a vector in \mathcal{N} and w :

$$x = \left(x - \frac{\phi(x)}{\phi(w)} w\right) + \frac{\phi(x)}{\phi(w)} w.$$

Uniqueness of the vector v is left as an exercise (not to be handed in).