Algebras Without Units

If $\mathcal{A}$ is a Banach algebra, it can be embedded isometrically as a subalgebra of a larger algebra.

$$\mathcal{A} \times \mathbb{C} = \hat{\mathcal{A}} = \{(x, \alpha)\}$$

- addition + multiplication by scalars are defined in the obvious way

$$\beta_1 (x_1, \alpha_1) + \beta_2 (x_2, \alpha_2) := (\beta_1 x_1 + \beta_2 x_2, \beta_1 \alpha_1 + \beta_2 \alpha_2)$$

- multiplication:

$$(x, \alpha) \cdot (y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

$$\|(x, \alpha)\|_{\hat{\mathcal{A}}} := \|x\|_{\mathcal{A}} + |\alpha|$$

verification of associativity:

$$((x, \alpha) \cdot (y, \beta)) \cdot (z, \gamma) =$$

$$(xy + \alpha y + \beta x, \alpha\beta) \cdot (z, \gamma) =$$

$$(xyz + \alpha yz + \beta xz + \gamma xy + \alpha\gamma y + \beta\gamma x + \alpha\beta z, \alpha\beta\gamma)$$

$$= (x, \alpha) \cdot ((y, \beta) \cdot (z, \gamma))$$

distributivity also holds

$\mathbb{1} \in \hat{\mathcal{A}}$? $\quad (x, \alpha) \cdot (0, 1) = (x \cdot 0 + \alpha \cdot 0 + 1 \cdot x, 1 \cdot \alpha) = (x, \alpha)$

The original algebra can be obtained from $\hat{\mathcal{A}}$ by a "projection" homomorphism

$\pi(x,\alpha) := (x, 0)$ $\qquad\qquad (x,0)\cdot(y,0) = (xy, 0)$

The subset $\{(x,0)\}$ of $\hat{\mathfrak{a}}$ is isomorphic to $\mathfrak{a}$

The complete definition of π requires consideration of quotient

Def: A homomorphism $\mathcal{A} \xrightarrow{\omega} \mathcal{B}$ is a linear map $\mathcal{A} \to \mathcal{B}$ $\ni$: $\omega(xy) = \omega(x)\,\omega(y)$. A Banach homomorphism also has the property that $\|\omega(x)\| \le \|x\|$ and $\|\omega(x)\,\omega(y)\| \le \|\omega(x)\|\|\omega(y)\|$. A C^*-algebra also has the property that $\omega(x^*) = (\omega(x))^*$

Eg. 3×3 diagonal matrices

$\omega(X) := x_{11}$

$\omega(\alpha X + \beta Y) = \alpha x_{11} + \beta y_{11}$

$\omega(XY) = x_{11}\, y_{11} = \omega(X)\,\omega(Y)$

Any isomorphism is a homomorphism

$K \in \mathcal{K}(\mathcal{H})$, $K = K^*$

$$C(\sigma(K)) \simeq \left\{ \sum_{n=1}^{\infty} f(\lambda_n) P_{\hat{e}_n} \right\}$$

$$\|f\|_\infty = \|f(K)\|_{op}$$

Ideals of $\mathcal{A}$

$\mathcal{A}$ is *simple* if the only ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$

An ideal is *proper* if it is not $\{0\}$ or $\mathcal{A}$

Claim: M^{nn} ($n \times n$ matrices) is simple

- It is possible for a subalgebra of a simple algebra to not be simple
  - diagonal $n \times n$ matrices in M^{nn} form a subalgebra

    $d_k := \{ D \mid D_{kk} = 0 \}$ is an ideal

    $D_1 \cdot D_2 \in d_k$ if D_1 or $D_2 \in d_k$

Suppose $\mathcal{I}$ is an ideal of M^{nn}, $\mathcal{I} \neq \{0\}$

$\exists M \in \mathcal{I} \ni: m_{kl} = 0$

Construct a sequence of operators $\ni: M \rightarrow \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}$

and a sequence of operators $\ni: M \rightarrow \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}$, etc.

If this is possible, $\mathcal{I} \ni \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} = \mathbb{1}$

A: $\hat{e}_1 \rightarrow \hat{e}_l$ $\qquad$ $MA\hat{e}_1 = l$ th column of M

$\hat{e}_k \rightarrow 0$ for $k \neq 1$

find $B \ni: BN = k$ th row of N, filled out with 0's

$C\left(\frac{1}{m_{kl}}\mathbb{1}\right) BMA\hat{e}_1 = \hat{e}_1$ $\qquad$ C interchanges $\hat{e}_k \leftrightarrow \hat{e}_1$

$C\left(\frac{1}{m_{kl}}\mathbb{1}\right) BMA\hat{e}_1 = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} \in \mathcal{I} \Rightarrow \mathbb{1} \in \mathcal{I} \Rightarrow \mathcal{I} = \mathcal{A}$ ($= M^{n}$

Lemma: If $\mathcal{I}$ is a proper ideal of a unital Banach algebra $\mathcal{A}$ and $\exists z \in \mathcal{I} \ni$: $\|\mathbb{1} + z\| \geq 1$, the closure of $\mathcal{I}$ is a proper ideal.

Proof: Suppose $\|\mathbb{1} + z\| < 1$. $z = (\mathbb{1} + z) - \mathbb{1} \Rightarrow z^{-1} \in \mathcal{A}$. Closure follows from continuity of the algebra operations