MATH 1502 EXAM #3 NAME: ______________________________

April 7, 2005

Answer the following questions clearly and completely. Unless otherwise specified, you must provide work justifying your solution. Use the back of the page for scratchwork if needed, but clearly indicate if anything on the back of the page should be graded, otherwise it will be ignored.

There are 3 questions plus one extra credit problem, on 7 pages. The exam is worth 40 points total, plus up to 3 points extra credit.

1. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 \\ 0 & -1 & 1 & 2 \\ 1 & 2 & -1 & 2 \end{bmatrix}$.

(8 points) a. Compute $\det(A)$ by using cofactors to reduce the 4×4 determinant to a sum of 3×3 determinants. You can then use any valid method you like to find those 3×3 determinants (if the scalar multiplying a particular 3×3 determinant is zero, you don't need to compute it).

(Problem 1 Continued)

(2 points) b. Is the matrix A invertible? How do you know?

(2 points) c. What is $\text{rank}(A)$? (No explanation required.)

2. Let T be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_3 + x_4 \\ 2x_1 + x_2 + 5x_3 + 4x_4 \\ -2x_1 - 3x_3 - x_4 \end{bmatrix}$.

(1 point) a. Give the domain and codomain of T (no explanation required).

(2 points) b. Give the standard matrix A for T (no explanation required).

(2 points) c. Explain why $\text{range}(T) = \text{Col}(A)$.

(Problem 2 Continued)

(5 points) d. Find a spanning set for $\mathrm{Null}(T)$.

(Problem 2 Continued)

(4 points) e. Find a spanning set for range(T).

(2 points) f. Is the spanning set for range(T) that you found in part c linearly independent? How do you know?

(Problem 2 Continued)

(2 points) g. Is T onto? How do you know?

(2 points) h. Is T 1-1? How do you know?

(2 points) i. Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a spanning set for range(T)? Why or why not?

(6 points) 3. For each part, circle T for True or F for False (no explanation required). Note: An answer of "sometimes true and sometimes false" or "cannot be determined" would count as False.

T F a. If a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is 1-1 and A is its standard matrix, then $Ax = b$ is consistent for every $b \in \mathbf{R}^m$.

T F b. The transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 + x_2 \\ -2x_1 \end{bmatrix}$ is linear.

T F c. If A is a square matrix and $\det(A) = 0$, then the reduced row echelon form of A has at least one zero row.

T F d. If A is an $m \times n$ matrix and $\mathrm{Null}(A) = \{0\}$, then the columns of A are linearly independent.

T F e. If W is a subspace of $\mathbf{R}^n$ then the standard vectors $e_1, \ldots, e_n$ are a spanning set for W.

T F f. $W = \left\{ \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} : a \in \mathbf{R} \right\}$ is a subspace of $\mathbf{R}^3$.

(3 points EXTRA CREDIT) Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that every vector in $\mathrm{Col}(AB)$ is also a vector in $\mathrm{Col}(A)$.