2.1 #42. a. We’re given that $v_1$ people live in the city and $v_2$ people live in the suburbs. We’re told that 60% of the city dwellers drive cars and 30% of the suburb dwellers drive cars. Therefore, the total number of people who drive cars is

$$u_1 = .6v_1 + .3v_2.$$  

Hence the first row of the matrix $B$ is going to be $[.6 \  .3]$, because we then have that

$$\begin{bmatrix} .6 & .3 \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ * \end{bmatrix}.$$  

We use similar reasoning to figure out how to compute $u_2$ and $u_3$ from $v_1$ and $v_2$, and end up with

$$B = \begin{bmatrix} .6 & .3 \\ .3 & .5 \\ .1 & .2 \end{bmatrix}.$$  

2.1 #54. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Hence rank$(AB) = 1$ while rank$(BA) = 0$.

2.3 #6. Since $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, we see that $B$ is NOT the inverse of $A$.

2.3 #32. We are given the 2nd, 4th, and 6th columns of $A$. That is, we know $a_2$, $a_4$, and $a_6$, and our task is to find $a_1$, $a_3$, $a_5$.

The second column of $R$ is twice the first column. Hence the same is true of $A$, i.e., $a_2 = 2a_1$. Now we know $a_1$.

The third column plus the fourth column of $R$ equals first column: $r_1 = r_3 + r_4$. Since the columns of $A$ have exactly the same relationships as the columns of $R$, this tells us that $a_1 = a_3 + a_4$. We already know $a_1$ and $a_4$, so this tells us $a_3$.

Finally, $r_4 + r_5 = r_6$, so $a_4 + a_5 = a_6$. Since $a_4$ and $a_6$ are known, this tells us $a_5$. 

Putting it all together, we find that \( A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & -1 & 3 & -4 & -1 \\ 3 & 6 & 4 & -1 & 0 & -1 \\ -1 & -2 & -2 & 1 & 1 & 2 \end{bmatrix} \).

2.4 #50. By definition, \( A \) similar to \( B \) means that \( B = P^{-1}AP \) for some invertible matrix \( P \). Since we’re told that \( A \) is invertible, \( B \) is therefore a product of invertible matrices and hence is itself invertible. Moreover, taking the inverse of both sides of \( B = P^{-1}AP \) tells us:
\[
B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P.
\]
Now, we’re hoping to show that \( A^{-1} \) is similar to \( B^{-1} \). That means we have to show that
\[
B^{-1} = Q^{-1}A^{-1}Q
\]
for some invertible matrix \( Q \). We’ve already shown that \( B^{-1} = P^{-1}A^{-1}P \). So the \( Q \) we need is \( Q = P \) (but there’s no reason that it HAS to be the same matrix, just that there is SOME invertible \( Q \) that works).

2.6 #14. We’re given that \( T \) is linear with \( T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( T\left(\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \), and we have to determine \( T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) \). Note that \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \). Since \( T \) is linear, we therefore have
\[
T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + T\left(\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}. \]

2.6 #28. Since \( T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), we have that \( T(0) \neq 0 \), so \( T \) cannot possibly be linear.

2.6 #52. \( T_A \) is the linear transformation defined by the rule \( T_A(x) = Ax \) for every \( x \in \mathbb{R}^n \). \( T_B \) is the linear transformation defined by the rule \( T_B(v) = Bv \) for every \( v \in \mathbb{R}^p \). \( T_{AB} \) is the linear transformation defined by the rule \( T_{AB}(v) = (AB)v \) for every \( v \in \mathbb{R}^p \). Hence
\[
T_{AB}(v) = (AB)v = A(Bv) = T_A(Bv) = T_A(T_B(v)).
\]

2.7 #6. The standard matrix is \( A = \begin{bmatrix} 5 & -4 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). The columns of \( A \) are a spanning set for \( \text{range}(T) \).
2.7 #23. Find a spanning set for the nullspace of $T: \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_2 + x_3 - x_4 \\ x_1 + x_2 + 2x_3 + 2x_4 \\ x_1 - x_3 - 3x_4 \end{bmatrix}.$$ 

**Solution**

The nullspace is the set of all vectors $x$ that satisfy $T(x) = 0$. So you just have to find the general solution to the system

$$2x_1 + x_2 + x_3 - x_4 = 0,$$
$$x_1 + x_2 + 2x_3 + 2x_4 = 0,$$
$$x_1 - x_3 - 3x_4 = 0.$$ 

Set up the matrix form of this system of linear equations and do the row reduction. The appropriate matrix $A$ and its row reduced echelon form are

$$A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & -1 & -3 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Therefore the general solution to $T(x) = 0$ in parametric form is

$$x = \begin{bmatrix} x_3 + 3x_4 \\ -3x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$ 

The nullspace is the SET of all these solutions, i.e.,

$$\text{Null}(T) = \left\{ x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

So the two vectors on the last line above are the spanning vectors that we seek. Note that the spanning set is just those two vectors, i.e., it is the set

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\},$$

while the nullspace itself is the set of infinitely many linear combinations of these two vectors. Note also that since those two vectors are independent (why?), their span is a plane. Thus the nullspace of this $T$ is a plane within $\mathbb{R}^4$. □