1. Let \( A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \).

(10 points) a. If \( A \) is invertible, find \( A^{-1} \).
(Problem 1 Continued)

(1 point) b. Using the matrix $A^{-1}$ you found in part a, compute $AA^{-1}$.

(3 points) c. Find elementary matrices $E_1, \ldots, E_k$ (how many depends on how you did part a on the preceding page) such that $E_k \cdots E_1 A = I$.

Hint: You’ve already done the work, just write them down. $I$ is the reduced row echelon form of $A$. 
2. Let \( W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 - x_2 + 3x_3 + x_4 = 0 \\ 3x_1 - x_2 + 5x_3 + 3x_4 = 0 \end{array} \right\} \)

(10 points) a. Is \( W \) a subspace of \( \mathbb{R}^4 \)? Why or why not? If it is, find a basis for \( W \) and find \( \dim(W) \).
(Problem 2 Continued)

(3 points) b. Find a basis for $W^\perp$ and find $\dim(W^\perp)$. 
3. Let \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 1 \end{bmatrix} \), and let \( W = \text{span}\{u_1, u_2\} \). Let \( v = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} \).

(2 points) a. What is \( u_1 \cdot u_2 \)? Are \( u_1, u_2 \) orthogonal vectors?

(10 points) b. Find the vector \( p \) in \( W \) that is closest to \( v \).
(Problem 3 Continued)

(2 points) c. Find vectors $p \in W$ and $e \in W^\perp$ such that $v = p + e$. 
4. Let \( A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 5 & 5 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix} \).

(6 points) a. Find the characteristic polynomial of \( A \).

(4 points) b. Find the eigenvalues of \( A \) and state the multiplicity of each. Based SOLELY on this information, can you determine whether \( A \) is invertible or not?
(Problem 4 Continued)

(10 points) c. Find a basis for each eigenspace of $A$. 
(Problem 4 Continued)

(8 points) d. If $A$ is diagonalizable, write down a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$. You do not need to find $P^{-1}$.

(4 points) e. Is $w = \begin{bmatrix} -4 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ an eigenvector of $A$? How do you know?
(4 points) 5. Suppose that $v_1$, $v_2$ are orthonormal vectors in $\mathbb{R}^n$. Find $\|v_1 - v_2\|$. 
Hint: Draw a picture.
6. Compute the following if possible, otherwise write *undefined* or *nonsense* as appropriate.

(1 point) a. A basis for the matrix
\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 2 \\
1 & 0 & 2
\end{bmatrix}
\]

(1 point) b. The determinant of
\[
\begin{bmatrix}
3 & 2 & 1 \\
0 & 2 & 2 \\
0 & 0 & 2
\end{bmatrix}
\]

(1 point) c. 
\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
1 & 0
\end{bmatrix}^T \begin{bmatrix}
2 & 0 & -1 \\
0 & 2 & 2
\end{bmatrix} =
\]

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7. Complete the following.

(1 point) a. Let $v_1, \ldots, v_k$ be $k$ vectors in $\mathbb{R}^n$. Define $\text{span}\{v_1, \ldots, v_k\}$.

(1 point) b. Vectors $v_1, \ldots, v_n$ in $\mathbb{R}^m$ are linearly independent if

(1 point) c. Let $A$ be an $m \times n$ matrix. Define $\text{Nul}(A)$. 

8. For each part, circle T for True or F for False (no explanation required). Note: An answer of “sometimes true and sometimes false” or “cannot be determined” would count as False.

T   F  a. If $A$ is an $7 \times 8$ matrix with 3 pivots, then the rowspace of $A$ is a 3-dimensional subspace of $\mathbb{R}^8$.

T   F  b. If $A$ is an $n \times n$ matrix and $\lambda$ is an eigenvalue of $A$, then $\dim(\text{Nul}(A - \lambda I_n)) \leq \text{multiplicity}(\lambda)$.

T   F  c. If $P$ is an invertible $n \times n$ matrix and $D$ is a diagonal matrix and if $A = PDP^{-1}$, then the diagonal entries of $D$ are the eigenvalues of $A$, and the columns of $P$ are the eigenvectors of $A$.

T   F  d. If $W$ is a subspace of $\mathbb{R}^n$ and $v \in W$, then the orthogonal projection of $v$ onto $W$ is 0.

T   F  e. The dot product of $u, v \in \mathbb{R}^n$ is $u \cdot v = uv^T$.

T   F  f. Eigenvalues of an $n \times n$ matrix $A$ must be nonzero, but eigenvectors can be zero.

T   F  g. If $v_1, \ldots, v_k$ are orthogonal vectors in $\mathbb{R}^n$, then they are linearly independent.

(3 points EXTRA CREDIT) Suppose that $A$ is an $n \times n$ matrix which has the property that $A^5 = 0$ (note that this does NOT imply that $A = 0$!). Prove that $\lambda = 0$ is the only eigenvalue of $A$. (Use back of the page for answer.)

(3 points EXTRA CREDIT) Prove that if $v_1, \ldots, v_k$ are orthonormal vectors in $\mathbb{R}^n$, then they are linearly independent. (Use back of the page for answer.)