Answer the following questions clearly and completely. Unless otherwise specified, you must provide work justifying your solution.

There are 5 questions plus one extra credit problem, on 7 pages. The exam is worth 40 points total, plus up to 3 points extra credit.

1. Compute the following quantities if possible, otherwise mark them as “undefined” (“nonsense” is the same as “undefined”).

   (1 point) a. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

   (1 point) b. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} =$

   (1 point) c. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} =$

   (1 point) d. $\text{span} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$
2. Let \( A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix} \) and \( b = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} \).

(4 points) a. Find the general solution to \( Ax = b \). Write your solution in parametric form.

(1 point) b. Give the reduced row echelon form \( R \) of \( A \) (no work required; you should have done it already in part a).
(2 points) c. Give elementary matrices $E_1, \ldots, E_k$ (how many depends on how you do the problem) such that $R = E_k \cdots E_1 A$.

(2 points) d. Do the columns of $A$ span $\mathbb{R}^3$? How do you know?

(2 points) e. Are the columns of $A$ linearly independent? How do you know?

(2 points) f. What is the rank and nullity of $A$?
(Problem 2 Continued)

(2 points) g. Is $b$ a linear combination of the columns of $A$? If so, give one specific linear combination of the columns that equals $b$.

(2 points) h. Is any column of $A$ a linear combination of the other columns? If so, give the specific linear combination.

(2 points) i. Can you tell FROM THE WORK DONE SO FAR whether the equation $Ax = \begin{bmatrix} 12 \\ -43 \\ 8 \end{bmatrix}$ has a solution? Why or why not?
(4 points) 3. Suppose that $A$ is an invertible $n \times n$ matrix. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{R}^n$. Let $u_1, \ldots, u_n$ be the columns of $A^{-1}$. If $1 \leq j \leq n$, show that $u_j$ is a solution to the equation $Ax = e_j$. 
4. Let \( A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).

(4 points) a. Find the inverse of \( A \), if it exists.

(1 point) b. Use your answer from part a to solve the equation \( Ax = b \).
5. For each part, circle T for True or F for False (no explanation required). Note: An answer of “sometimes true and sometimes false” or “cannot be determined” would count as False.

T  F  a. If $A$ is an $m \times n$ matrix and $n > m$ then $Ax = 0$ has infinitely many solutions.

T  F  b. If $A$ is an $m \times n$ matrix and $n > m$ then for every $b \in \mathbb{R}^m$, the equation $Ax = b$ has infinitely many solutions.

T  F  c. If $A$ is an $m \times n$ matrix and $n < m$ then there is some $b \in \mathbb{R}^m$ such that the equation $Ax = b$ has no solutions.

T  F  d. If $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then the equation $Ax = b$ is consistent if and only if $b$ is a linear combination of the columns of $A$.

T  F  e. If vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly dependent, then at least one $v_j$ must be a multiple of one of the others.

T  F  f. If $E$ is an $n \times n$ elementary matrix, then the equation $Ex = b$ has at most one solution for each $b \in \mathbb{R}^n$.

T  F  g. If $u, v, w$ are independent vectors in $\mathbb{R}^n$ and if $x \in \text{span}\{u, v, w\}$, then $u, v, w, x$ are independent vectors in $\mathbb{R}^n$.

(3 points EXTRA CREDIT) Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix, with $n > m$. Prove that $AB$ can never equal the $m \times m$ identity matrix $I_m$. 
