Answer the following questions clearly and completely. Unless otherwise specified, you must provide work justifying your solution.

There are 5 questions plus one extra credit problem, on 8 pages. The exam is worth 40 points total, plus up to 3 points extra credit.

1. Let $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix}$.

   (6 points) a. Compute $\det(A)$ by using cofactors to reduce the $4 \times 4$ determinant to a sum of $3 \times 3$ determinants. You can then use any valid method you like to find those $3 \times 3$ determinants (if the scalar multiplying a particular $3 \times 3$ determinant is zero, you don’t need to compute it).
(Problem 1 Continued)

(2 points) b. WITHOUT doing row reduction, give the reduced row echelon form for $A$. Hint: This follows immediately from part a. You do NOT need to do any more work to find it.

(2 points) c. What is $\text{Col}(A)$? Hint: This follows immediately from part b.
(4 points) 2. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ x_1 x_2 \\ x_1 + x_2 \end{bmatrix}$.

Determine (with proof) whether $T$ is linear.
3. Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix. Which one of the following statements is true, and why?

A. If $x$ is in $\text{Null}(A)$, then $x$ is in $\text{Null}(AB)$.
B. If $x$ is in $\text{Null}(B)$, then $x$ is in $\text{Null}(AB)$.
C. If $x$ is in $\text{Null}(AB)$, then $x$ is in $\text{Null}(A)$.
D. If $x$ is in $\text{Null}(AB)$, then $x$ is in $\text{Null}(B)$. 
4. Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by $T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ x_2 + x_3 \\ 0 \\ x_1 + x_2 + 3x_3 \end{bmatrix}$.

(2 points) a. Find the standard matrix for $T$ (no explanation required).

(5 points) b. Find a basis for range($T$).
(Problem 4 Continued)
(1 point) c. Find the dimension of range($T$).

(5 points) d. Find a basis for Null($T$).
(Problem 4 Continued)

(2 points) e. Is $T$ 1-1? Why or why not?

(2 points) f. Is $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ a basis for $\text{Null}(T)$? Why or why not?
(5 points) 5. For each part, circle T for True or F for False (no explanation required). Note: An answer of “sometimes true and sometimes false” or “cannot be determined” would count as False.

T  F  a. The column space of an $m \times n$ matrix equals the set \{Ax : x \in \mathbb{R}^n\}.

T  F  b. If $A$ is an $m \times n$ matrix then $\text{rank}(A^T) = \dim(\text{Col}(A))$.

T  F  c. The columns of an $m \times n$ matrix form a basis for its column space.

T  F  d. If $W$ is a subspace of $\mathbb{R}^n$ and $u_1, \ldots, u_k$ is a spanning set for $W$, then there are vectors $u_{k+1}, \ldots, u_\ell$ in $\mathbb{R}^n$ such that $u_1, \ldots, u_k, u_{k+1}, \ldots, u_\ell$ is a basis for $W$.

T  F  e. There is only one subspace of $\mathbb{R}^5$ that has dimension 5.

(3 points EXTRA CREDIT) Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Prove that $\text{rank}(AB) \leq \text{rank}(B)$. Hint: Use Problem 3.