Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together. Remember that ALL STATEMENTS REQUIRE PROOF.

Definition. Let $V$ and $W$ be vector spaces and let $T: V \rightarrow W$ be given. We say that $T$ is linear or that $T$ is a linear transformation if

(a) $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$, and

(b) $T(cx) = cT(x)$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

1. Let $u, v, w, z$ be four nonzero vectors in $\mathbb{R}^n$. Define a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T(x) = (v \cdot x)u + (z \cdot x)w$, where $x \cdot y$ is the dot product $x \cdot y = x_1y_1 + \cdots + x_ny_n$.

(a) Prove that $T$ is linear.

Note: You can use facts about the dot product from Chapter 1 without proof. For example, the distributive rule holds for the dot product.

(b) Prove that $\text{range}(T) \subseteq \text{span}\{u, w\}$.

(c) Give an example of nonzero vectors $u, v, w, z \in \mathbb{R}^2$ for which $\text{range}(T) \neq \text{span}\{u, w\}$.

2. (a) Let $V, W$ be vector spaces and let $T: V \rightarrow W$ be a 1-1 linear transformation. Prove that if $v_1, \ldots, v_n$ are independent vectors in $V$, then $T(v_1), \ldots, T(v_n)$ are independent vectors in $W$.

Hint: The first line of your proof will be “Suppose that $c_1T(v_1) + \cdots + c_nT(v_n) = 0$ for some scalars $c_1, \ldots, c_n$.”

(b) Is the converse result also true? That is, if $T$ maps independent vectors to independent vectors, must $T$ be 1-1?

Hint: Think about $T(x) - T(y)$.

3. Let $V, W$ be finite-dimensional vector spaces with $\text{dim}(V) > \text{dim}(W)$. Let $T: V \rightarrow W$ be a linear transformation. Prove that $T$ cannot be 1-1.

Hint: Proceed by contradiction, i.e., suppose that $T$ was 1-1 and derive a contradiction from this by making use of Problem 2.
4. Let $\mathcal{P}_n$ be the space of all polynomials of degree at most $n$. Let
\[ S_n = \{ p \in \mathcal{P}_n : p(1) = 0 \}, \]
i.e., $S_n$ is the set of all polynomials of degree at most $n$ that have $x = 1$ as a root.

(a) Prove that $S_n$ is a subspace of $\mathcal{P}_n$.

(b) Compute a basis $B$ for $S_n$ and find the dimension of $S_n$.
Hint: If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $p(1)$ is what number?

(c) Let $n = 5$ and set $q(x) = (x - 1)^2 (x^2 + x + 1)$. Show that both $q$ and $q'$ are in $S_5$.

(d) Let $\mathcal{E} = \{1, x, x^2, x^3, x^4, x^5\}$ be the standard basis for $\mathcal{P}_5$. Compute $[q]_\mathcal{E}$ and $[q']_\mathcal{E}$, the coordinates of $q$ and $q'$ with respect to the standard basis. Also compute $[q]_B$ and $[q']_B$, the coordinates of $q$ and $q'$ with respect to the basis $B$ for $S_5$ that you found in part (b).