Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together. Remember that ALL STATEMENTS REQUIRE PROOF.

The space $C[a,b]$ consists of all continuous functions $f: [a, b] \to \mathbb{R}$. The inner product on $C[a,b]$ is $\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$.

The corresponding induced norm is the “$L^2$-norm” given by

$$
\|f\|_2 = \langle f, f \rangle^{1/2} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}.
$$

The “$L^1$-norm” on $C[a,b]$ is $\|f\|_1 = \int_a^b |f(x)| \, dx$.

Some useful facts: $\int_0^{2\pi} \sin^2 x \, dx = \int_0^{2\pi} \cos^2 x \, dx = \pi$.

1. (a) Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space $V$. Prove the Parallelogram Law:

$$
\forall u, v \in V, \quad \|u + v\|^2 + \|u - v\|^2 = 2 (\|u\|^2 + \|v\|^2).
$$

(b) Show that the $L^1$-norm on $C[0,1]$ does not satisfy the Parallelogram Law. That is, show that the following statement is false:

$$
\forall f, g \in C[0,1], \quad \|f + g\|_1^2 + \|f - g\|_1^2 = 2 (\|f\|_1^2 + \|g\|_1^2).
$$

(c) Show that the $L^1$-norm on $C[0,1]$ is not induced from any inner product. That is, show that there is no inner product $\langle \cdot, \cdot \rangle$ on $C[0,1]$ such that $\|f\|_1 = \langle f, f \rangle^{1/2}$ for all $f \in C[0,1]$.

2. Everything in this problem uses the inner product on $C[0,2\pi]$ and the corresponding induced norm (the $L^2$-norm). Define functions $f_1, f_2, f_3, g \in C[0,2\pi]$ by the rules

$$
 f_1(x) = 1, \quad f_2(x) = \sin x, \quad f_3(x) = \cos x, \quad g(x) = x.
$$

Let $U = \text{span}\{f_1, f_2, f_3\}$ and $V = \text{span}\{f_1, f_2, f_3, g\}$.

(a) Find the orthogonal projection $p$ of $g$ onto $U$.

(b) Find the function $q$ in $U$ that is closest to $g$.

(c) Plot (either by hand or using graphing software) the functions $g$ and $q$ on the same graph. Explain in words and in terms of this picture what it means for $q$ to be “the function in $U$ that is closest to $g$.”
(d) Find an orthonormal basis for $V$.

(e) Show that $V$ is isomorphic to $\mathbb{R}^4$. That is, find a map $T: V \to \mathbb{R}^4$ that is a linear bijection that is also norm-preserving, i.e.,

$$\forall f \in V, \quad \|T(f)\| = \|f\|_2.$$  

Note that $T(f)$ is a vector in $\mathbb{R}^4$ and $\|T(f)\|$ is the ordinary vector length of $T(f) \in \mathbb{R}^4$. 