INSTRUCTIONS.
Work the following problems and hand in your solutions. Your solutions must be TYPED. You may work together with other people in the class, but you must each write and type your solutions independently. Please LIST all people that you collaborated with. A subset of these problems will be selected for grading.

Definition 1. Let \( f: \mathbb{R} \to \mathbb{R} \) be a given function, and let \( x \in \mathbb{R} \) be a real number. We say that \( f \) has a limit at \( x \) if there is a number \( L \) such that
\[
\forall \varepsilon > 0, \quad \exists \delta > 0, \quad 0 < |x - y| < \delta \quad \Rightarrow \quad |f(y) - L| < \varepsilon.
\]
In this case, we write
\[
\lim_{y \to x} f(y) = L.
\]

1. Prove the following theorem.

Theorem 2. Suppose functions \( f \) and \( g \) each have a limit at \( x \), and let \( L = \lim_{y \to x} f(y) \) and \( M = \lim_{y \to x} g(y) \).

Given real numbers \( a, b \in \mathbb{R} \), let \( h \) be the function whose rule is
\[
h(t) = af(t) + bg(t), \quad t \in \mathbb{R}.
\]
Then \( h \) has a limit at \( x \), and this limit is
\[
\lim_{y \to x} h(y) = aL + bM.
\]

Proof. Type the proof here. \( \square \)

2. Given \((a, b) \in \mathbb{R}^2\), define a function \( g_{ab}: \mathbb{R} \to \mathbb{R} \) by the rule
\[
g_{ab}(x) = ax + b, \quad x \in \mathbb{R}.
\]
(a) Determine (with proof) all values of \((a, b) \in \mathbb{R}^2\) for which \( g_{ab} \) is injective.
(b) Determine (with proof) all values of \((a, b) \in \mathbb{R}^2\) for which \( g_{ab} \) is surjective.
(c) Determine (with proof) all values of \((a, b) \in \mathbb{R}^2\) for which \( g_{ab} \) is bijective.
3. Let $\mathcal{F}(\mathbb{R})$ be the set of all functions that map real numbers to real numbers, i.e.,

$$\mathcal{F}(\mathbb{R}) = \{ f : f : \mathbb{R} \to \mathbb{R} \}.$$  

Define a function $\delta : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ by the rule

$$\delta(f) = f(\pi), \quad f \in \mathcal{F}(\mathbb{R}).$$

(a) If $f(x) = \sin x$, what is $\delta(f)$?

(b) Determine (with proof) whether $\delta$ is injective.

(c) Determine (with proof) whether $\delta$ is surjective.

(d) Determine (with proof) whether $\delta$ is bijective.

4. Using the same notation as in Problem #2, define

$$A = \{ g_{ab} : (a, b) \in \mathbb{R}^2, a \neq 0 \}. $$

Define a function $G : \mathbb{R}^2 \to A$ by the rule

$$G((a, b)) = g_{ab}, \quad (a, b) \in \mathbb{R}^2, a \neq 0.$$  

Determine (with proof) whether $G$ is injective, surjective, or bijective.