To help you prepare for our first exam, work some or all of the problems below. This problem set covers material from the online “Chapter 1” lecture notes (there will be no questions on the exam from Sections 1.6 or 1.7 of the lecture notes). *Write complete and careful proofs!*

1. Prove that $\sqrt{5}$ is irrational.

2. (a) Prove that the product of two rational numbers is rational.
   (b) Is the product of two irrational numbers always irrational? Either prove that it is or find a counterexample.

3. Write the truth tables for the following statements.
   (a) If $A$ then $(B$ or $C)$.
   (b) NOT $(A$ implies $B)$.
   (c) $A$ or $B$ and $(NOT \ C)$.
   (d) $A$ if and only if $(B$ and $C)$.

4. (a) Using the definition we gave in class, complete the following: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a point $x$ if
   (b) Write the negation of the definition given in part (a). In other words, complete the following: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is not continuous at a point $x$ if

7. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function whose rule is $f(x) = x^3$ for $x \in \mathbb{R}$. Prove that $f$ is continuous at the point $x = 7$.

6. The Dirichlet function is

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let $x$ be any real number. Prove that $f$ is not continuous at $x$. 
7. Challenge problem (not easy!!). Define a function $g : \mathbb{R} \to \mathbb{R}$ in the following way. If $x$ is a nonzero rational number and we write $x$ in lowest terms as $x = p/q$, then we set

$$g(x) = \frac{1}{q}.$$ 

If $x = 0$ or $x$ is irrational, then we set

$$g(x) = 0.$$ 

(a) Prove that if $x$ is rational, then $g$ is not continuous at $x$ (easy).

(b) Prove that if $x$ is irrational, then $g$ is continuous at $x$ (harder!).

8. Prove that if functions $f$ and $g$ are each continuous at a point $x$, then the function $h = f + g$ is also continuous at $x$.

9. Let $x$ be a fixed real number. Suppose that $f_1, f_2, \ldots$ are functions that are each continuous at $x$. Given any integer $N$, prove that the function $g = f_1 + \cdots + f_N$ is continuous at $x$.

Hint: Use induction.

10. State the definition of a limit that we gave in class. That is, complete the following:

A function $f : \mathbb{R} \to \mathbb{R}$ has a limit at the point $x$ if there is a number $L$ such that

$$\lim_{y \to x} f(y) = L.$$ 

11. Suppose that $f$ and $g$ are functions that each have a limit at a point $x$. To make things a bit easier, let us also suppose that $f$ and $g$ are both bounded functions (what exactly does this mean?). Prove that the function $h = fg$ has a limit at $x$, and we have

$$\lim_{y \to x} h(y) = \left( \lim_{y \to x} f(y) \right) \left( \lim_{y \to x} g(y) \right).$$ 

Hint: Give things some names, let $L = \lim_{y \to x} f(y)$ and $M = \lim_{y \to x} g(y)$. Your given information is that $L$ and $M$ exist, and your task is to show that $h$ has a limit at $x$ and that limit is the number $LM$.

12. Given a function $f : \mathbb{R} \to \mathbb{R}$ and given a point $x \in \mathbb{R}$, prove that

$$f \text{ is continuous at } x \iff \lim_{y \to x} f(y) = f(x).$$ 

Note: Writing “$\lim_{y \to x} f(y) = f(x)$” is shorthand for the more complete statement “$f$ has a limit at $x$ and that limit is $\lim_{y \to x} f(y) = f(x)$”.

13. Is $\mathbb{Z} \subseteq \mathbb{R}$? Is $\mathbb{Z} \subset \mathbb{R}$?
14. Let $X$ be a set, and for each $k \in \mathbb{N}$ let $A_k$ be a subset of $X$. Also let $B$ be a subset of $X$. Prove the following statements. (What do the symbols $\bigcup$ and $\bigcap$ mean? What does $S^C$ mean?)

(a) \( \left( \bigcup_k A_k \right) \cup B = \bigcup_k (A_k \cup B) \).

(b) \( \left( \bigcup_k A_k \right) \cap B = \bigcup_k (A_k \cap B) \).

(c) \( \left( \bigcap_k A_k \right) \cup B = \bigcap_k (A_k \cup B) \).

(d) \( \left( \bigcap_k A_k \right) \cap B = \bigcap_k (A_k \cap B) \).

(e) \( \left( \bigcap_k A_k \right)^C = \bigcup_k A_k^C \).

(f) \( \left( \bigcup_k A_k \right)^C = \bigcap_k A_k^C \).

15. Determine whether each of the following sets is bounded above, bounded below, bounded, or unbounded.

(a) \( \bigcup_{k \in \mathbb{Z}} [2k, 2k + 1] \).

Remark: The notation $[a, b]$ means a closed interval: $[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$.

(b) $\mathbb{Q} \setminus [0, 1]$.

(c) $[0, 1] \setminus \mathbb{Q}$.

(d) $\emptyset$.

16. Let $S$ be a subset of $\mathbb{R}$. Prove that if $S$ is bounded above, then there are infinitely many numbers that are upper bounds for $S$.

17. Let $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$. Prove that $\sup(S) = 1$ and $\inf(S) = 0$. (Can you give the precise definition of $\sup$ and $\inf$ from memory?)

18. Let $S$ be a set of real numbers that is bounded above, and let $T = \{ 5x : x \in S \}$. Prove that $T$ is bounded above. Also, find (with proof) the relationship between $\sup(S)$ and $\sup(T)$. 
Let $S$ be a bounded set of real numbers (what is the difference between bounded and bounded above?). Let

$$U = \{-x : x \in S\}.$$ 

Prove that $U$ is bounded, and we have

$$\sup(U) = -\inf(S).$$

20. Suppose that $A = \{a_k : k \in \mathbb{N}\}$ and $B = \{b_k : k \in \mathbb{N}\}$ are sets of real numbers that are each bounded above. Prove that

$$C = \{a_k + b_k : k \in \mathbb{N}\}$$

is bounded above, and

$$\sup(C) \leq \sup(A) + \sup(B).$$

Must there actually be equality on the line above? If so, prove this, and if not, give a counterexample.

21. (Challenge Problem) Let $S$ be a set of real numbers that is bounded above. Prove that the following two statements are equivalent.

(a) $u = \sup(S)$.

(b) $u$ is an upper bound for $S$, and for every $\varepsilon > 0$, there is a number $x \in S$ such that $u - \varepsilon < x \leq u$.

Remark: Saying that “(a) and (b) are equivalent” is the same as saying that “(a) is true if and only if (b) is true.” So, you have to do two proofs to solve this problem: first you have to show that if (a) is true then (b) is true, and then you have to prove that if (b) is true then (a) is true.

22. Assume that $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ are each bounded above. Prove that the set

$$Z = \{x + y : x \in X, y \in Y\}$$

is bounded above, and

$$\sup(Z) = \sup(X) + \sup(Y).$$

Hint: You need to use Problem 21.

23. (a) Let $S$ be a bounded set of real numbers, and let

$$S^2 = \{x^2 : x \in S\}.$$ 

Prove that $S^2$ is bounded. Is it true that we must have

$$\sup(S^2) = \sup(S)^2?$$

Either prove that this is true, or find a counterexample.

(b) Same question, except now assume that $S$ is a bounded set of positive real numbers.