1.6 Russell's Paradox

We said that an informal definition is "a set is a collection of objects." This isn't really precise, it essentially says that "a set is a set of objects," which doesn't define what a set is. For most purposes, it's not worth the trouble of trying to say exactly what a set is -- take a course in set theory for the details. But it is fun to point out what the "problem" with sets is.

Let

\[ G = \{ S : S \text{ is a set } \& S \not\in S \} \]

That is, \( G \) is the set of all sets that are not elements of themselves. For example, the set of all real numbers is not itself a real number, so \( \mathbb{R} \not\in \mathbb{R} \) and therefore \( \mathbb{R} \) is one of the elements of \( G \).

Let

\[ B = \{ S : S \text{ is a set } \& S \in S \} \]

I confess that I don't know of any sets that are elements of themselves, so I can't show you any elements of \( B \), but who knows, maybe some exist. If they don't, then \( B \) is simply the empty set.
The letter $G$ is for "good" & $B$ is for "bad."

Every set $S$ is either good or bad: we must have either $S \in S$ or $S \notin S$ (and in this case, it's an exclusive or, we can't have both). So if $G$ is a set then there are only two possibilities:
$G \notin G$ or $G \in G$.

If $G \in G$ then $G$ is a set & $G \notin G$, so $G$ satisfies the requirements to be in $G$. That is, $G$ is one of the good sets, i.e., $G \in G$. CONTRADICTION!

If $G \notin G$ then $G$ has to satisfy the requirements of elements of $G$, which are that $G$ is a set and $G \notin G$.

CONTRADICTION.

Either possibility leads to a contradiction. $G$ being a set leads to a contradiction. Only way out:

$G$ is not a set.

Moral: You can't just write \( \{ x : x \text{ satisfies property } P \} \) and always get a set. However, in practice these exceptions are very rare.