2.2 Compositions

Definition

Let $A, B, C$ be sets. The composition of a function $g : B \to C$ with a function $f : A \to C$ is the function $gof : A \to C$ given by the rule

$$(gof)(x) = g(f(x)), \quad x \in A.$$
Note: To prove $gof$ is surjective, you have to prove that range $(gof) = C$. A picture is not proof. How do you prove that two sets are equal?

d. Give an example of functions $f$, $g$ such that $gof$ is surjective even though $f$, $g$ are not both surjective.

Fill in the blank: Part d of this exercise shows that

De _______ of part c is false.

Exercise

a. Prove that if $f$ & $g$ are both injective, then $gof$ is injective.

b. Is the converse of part a true? Either prove it or give a counterexample.

c. Prove that if $f$ & $g$ are both bijections, then $gof$ is a bijection.

Note: Don't make things hard on yourself. If you've already done d. exercises on injectivity & surjectivity given before, then this is a one-line proof.

d. Converse of part c?
Example
Recall our translation function $T_a : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

Given a function $f \in \mathcal{F}(\mathbb{R})$, the function $T_a$ takes the input $f$ and outputs a new function $T_a f$ whose rule is

$$(T_a f)(x) = f(x-a), \quad x \in \mathbb{R}.$$ 

Consider now two values of the translation parameter. That is, choose $a, b \in \mathbb{R}$ and consider two functions $T_a : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ and $T_b : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$. We can compose $T_a$ & $T_b$ and get a new function $T_a \circ T_b : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

What is this composition $T_a \circ T_b$? That is, what is its rule? Let's work it out. Given an element $f$ of the domain $\mathcal{F}(\mathbb{R})$, the definition of composition says that

$$(T_a \circ T_b)(f) = T_a(T_b(f)).$$

What kind of object is $f$? Number? Function? Apple?

It's an element of $\mathcal{F}(\mathbb{R})$ — it's a function that maps real numbers to real numbers.

Let's give $T_a$ some names, hopefully this
will simplify \& notation. We have a function \((T_a \circ T_b)(f)\) \& we know it equals the function \(T_a(T_b(f))\):

\[
(T_a \circ T_b)(f) = T_a(T_b(f))
\]

Let's just call \(T_b(f) = g\)

Let's call this part \(h\).

Our goal is to figure out what \(g\) is. What we know is that

\(g = T_a h\) where \(h = T_b f\). So let's work out the rule for \(g\). Given a real number \(x\) we have

\[g(x) = T_a h(x) = h(x-a)\]

But we know that \(h = T_b f\), so let's use that info:

\[g(x) = h(x-a) = (T_b f)(x-a) = f((x-a)-b)\].
In summary,
\[ g(x) = f(x - a - b). \]

Thus \( g \) is a translation of \( f \) by \( d \) amount \( a+b \):
\[ g(x) = f(x-(a+b)) = T_{a+b}f(x). \]

Since \( g \) and \( T_{a+b}f \) have the same rule, what we've shown is that \( g \) & \( T_{a+b}f \) are the same function:
\[ g = T_{a+b}f. \]

Now remember what \( g \) was:
\[ (T_{a} \circ T_{b})(f) = g = T_{a+b}(f). \]

That is, given any input function \( f \), the functions \( T_{a} \circ T_{b} \) and \( T_{a+b} \) have the same output \( g \). Hence these two functions are identical:
\[ T_{a} \circ T_{b} = T_{a+b}. \]

Translation by \( a \) composed with translation by \( b \)
is by some a translation by \( a + b \). This is precisely what we expect, but we have proved it rigorously.

We can write all this in a much shorter form. See if you understand & believe the following proof.

**Theorem**

If \( a, b \in \mathbb{R} \) then \( T_a \circ T_b = T_{a+b} \).

**Proof:**

To show that \( T_a \circ T_b \) and \( T_{a+b} \) are the same function, we must show that they have the same rule. Therefore, here is our goal:

**Goal:** Show that \( (T_a \circ T_b)(f) = T_{a+b}(f) \)

for every \( f \in F(\mathbb{R}) \).

Now, to prove that \( (T_a \circ T_b)(f) \) & \( T_{a+b}(f) \) are the same function, we have to prove that they have the same rule. So, let \( x \in \mathbb{R} \) be any point in the domain of these functions. Then we have:
\[
(T_a \circ T_b)(f)(x) \\
= \left( T_a \left( T_b(f) \right) \right)(x) \quad \text{by def of composition} \\
= \left( T_b(f) \right)(x-a) \quad \text{by def of } T_a \\
= f \left( (x-a) - b \right) \quad \text{by def of } T_b \\
= f \left( x - (a+b) \right) \\
= \left( T_{a+b} f \right)(x) \quad \text{by def of } T_{a+b}.
\]

Or, if you’re ok with omitting some parentheses, it might be more clear to write

\[
(T_a \circ T_b)(f)(x) = \left( T_a \left( T_b f \right) \right)(x) \\
= T_b f (x-a) \\
= f \left( (x-a) - b \right) \\
= T_{a+b} f (x).
\]

Either way, we’ve shown that \( (T_a \circ T_b)(f) \) and \( T_{a+b} f \) have the same rule, so they’re also the same function. Hence we’ve proved our goal.
Notation Warning.

You may be tempted to write

\[(T_a o T_b)(f))(x) = T_b(f(x)).\]

But this makes NO SENSE.

It is WRONG to write

\[(T_b f)(x) = T_b(f(x)).\]

This is really shorthand for \((T_b(f))(x)\).

If is plugged into \(T_b\), which gives us a new function \(T_b(f)\), and \(x\) is plugged into that function.

The domain of \(T_b\) is \(F(\mathbb{R})\).

You plug a function into \(T_b\) and you get out a new function. But what kind of object is \(f(x)\)?

It is a number,

the value of \(f\) at \(x\).

\(f\) is a function,

\(f(x)\) is a number.
Exercises

Let $A, B, C$ be sets, and let $g : B \to C$ and $f : A \to B$ be functions.

(a) Show that if $g \circ f$ is injective, then $f$ is injective.

Did you do this problem already? (No!)

Must $g$ also be injective?

(b) Show that if $g \circ f$ is surjective, then $g$ is surjective.

Must $f$ be surjective?

(c) If $g \circ f$ is a bijection, must $f$ or $g$ be bijections?

Exercise

Let $A$ be a set & let $f : A \to A$ and $g : A \to A$ be functions. Is composition commutative?

That is, must $g \circ f = f \circ g$? Either prove or give a counter-example.

Exercise

Let $A, B, C, D$ be sets. Let

$f : A \to B$, $g : B \to C$, $h : C \to D$

be functions. Prove that composition is
associative, i.e.,

\[ h \circ (g \circ f) = (h \circ g) \circ f. \]

Hint: How do you prove that two functions are equal?