2.7 The Homomorphism Theorems

Motivation
Consider the map

\[ f: \mathbb{R}^2 \to \mathbb{R} \]
\[ f(a, b) = b \]

\( f \) sends a point \((a, b) \in \mathbb{R}^2\) to the number \(b \in \mathbb{R}\).

If we think of \(\mathbb{R}^2\) and \(\mathbb{R}\) as being groups under addition, then \(f\) is a homomorphism because

\[
f((a_1, b_1) + (a_2, b_2)) = f((a_1 + a_2, b_1 + b_2)) = b_1 + b_2 = f((a_1, b_1)) + f((a_2, b_2)).
\]
$f$ is surjective, but it is not injective — any two points with the same second coordinate are mapped to the same place. In fact,

$$N = \ker(f) = \{ (x,0) : x \in \mathbb{R} \} = \text{x-axis in } \mathbb{R}^2.$$ 

Recall that we earlier used $\mathbb{R}^2/N$ as an example motivating the definition of the quotient group. The cosets of $N$ are

$$N + (a,b) = N + (a,0) = \{ (x,0) : x \in \mathbb{R} \} = L_b,$$

the line a height $b$. The quotient group $\mathbb{R}^2/N$ is the set of all cosets of $N$:

$$\mathbb{R}^2/N = \{ L_b : b \in \mathbb{R} \},$$

the set of all horizontal lines in $\mathbb{R}^2$.

The canonical projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/N$ sends a point $(a,b)$ to the coset $N + (a,b) = L_b ;$
\[ \pi : \mathbb{R}^2 \to \mathbb{R}^2 \div N \]
\[ \pi((a,b)) = L_b \]

\( \pi \) is a homomorphism — we saw that the operation in \( \mathbb{R}^2 \div N \) is given by \( L_b + L_c = L_{b+c} \), so

\[ \pi((a_1,b_1) + (a_2,b_2)) = \pi((a_1+a_2,b_1+b_2)) \]

\[ = L_{b_1+b_2} \]

\[ = L_{b_1} + L_{b_2} \]

\[ = \pi((a_1,b_1)) + \pi((a_2,b_2)) \]
So now we have two maps on $\mathbb{R}^2$:

$$
\begin{array}{rcl}
\mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} \\
\text{surjective homomorphism} & & \\
\downarrow & & \\
\mathbb{R}^2/N & \xrightarrow{g} & \\
\text{surjective homomorphism} & & \\
\end{array}
$$

These two maps are related by the fact that

$N = \ker(f)$ and $\mathbb{R}^2$ maps onto $\mathbb{R}^2/N$.

The kernel of $f$ is used to make the quotient group $\mathbb{R}^2/N$.

Now compare the two ranges $\mathbb{R}$ and $\mathbb{R}^2/N$.

They are actually very similar!
\[ \mathbb{R} = \text{set of numbers} \quad \mathbb{R}^2/N = \text{set of lines} \]
\[ = \{ b : b \in \mathbb{R} \} \quad = \{ L_b : b \in \mathbb{R}^2 \} \]

Operation is
\[ b + c = \text{usual sum of } b \text{ and } c \]

Operation is
\[ L_b + L_c = L_{b+c} \]

These groups have the same structure, and they are actually isomorphic. That is, we claim that
\[ \mathbb{R}^2/N \cong \mathbb{R} \]

To prove this, we must show that there exists an isomorphism \( \Psi \) that maps \( \mathbb{R}^2/N \) to \( \mathbb{R} \).

Define
\[ \Psi : \mathbb{R}^2/N \to \mathbb{R} \]
\[ \Psi(L_b) = b \]

\( \Psi \) maps the line at height \( b \) to the number \( b \).
\( \mathbb{R}^3 / \mathbb{N} \)
Set of all lines \( L_b \)

\( \psi \) is a bijection, & it is a homomorphism because

\[
\psi(L_b + L_c) = \psi(L_{b+c}) = b + c = \psi(L_b) + \psi(L_c).
\]

This is a general phenomenon!!

\[
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\downarrow \text{surjective homomorphism} & & \downarrow \text{Then } \exists \text{ isomorphism } \psi \\
G / N & , & N = \ker(f)
\end{array}
\]
The following result is known by many names, including:

First Homomorphism Theorem
First Isomorphism Theorem
The Homomorphism Theorem
The Isomorphism Theorem

**First Homomorphism Theorem**

Let \( f : G \rightarrow H \) be a **surjective** homomorphism of \( G \) onto \( H \), and let \( N = \ker(f) \). Then

\[
G/N \cong H
\]

The isomorphism is

\[
\Psi : G/N \rightarrow H \quad \Psi(\alpha N) = f(\alpha)
\]

Further,

\[
f = \Psi \circ \pi
\]
Proof,

We have a proposed definition of the isomorphism $\Psi$ from $G/N$ to $H$, namely, $\Psi(Na) = f(a)$. But does this definition make any sense? We could have $Na = Nb$ even though $a \neq b$. In this case $\Psi(Na)$ and $\Psi(Nb)$ must have the same definition, but should they be defined to be $f(a)$ or $f(b)$? The only way that this definition will be OK is if

$$Na = Nb \implies f(a) = f(b).$$

But we don't know that this is true, we have to prove it. To do this, suppose that we had $Na = Nb$. Then $ab^{-1} \in N = \ker(f)$. Therefore

$$\Psi_H = f(ab^{-1}) = f(a) - f(b)^{-1}.$$  

because $ab^{-1} \in \ker(f)$ because $f$ is a homomorphism.
Hence we do indeed have \( f(a) = f(b) \). Therefore the proposed function \( \Psi \) is well-defined, i.e., it makes sense.

Now we have to show that \( \Psi \) actually is an isomorphism. First let's show that it is a homomorphism. Given \( Na, Nb \in G/N \), we have

\[
\Psi((Na)(Nb)) = \Psi(N(ab)) \quad \text{def. of group } \Psi \text{ in } G/N
\]
\[
= f(ab) \quad \text{def. of } f
\]
\[
= f(a)f(b) \quad \text{since } f \text{ is a homomorphism}
\]
\[
= \Psi(Na)\Psi(Nb) \quad \text{def. of } f.
\]

Thus \( \Psi \) is indeed a homomorphism.

Since \( \Psi \) is a homomorphism, to show it is injective, we just have to show that \( \ker(\Psi) \) contains only the identity element of \( G/N \), which is...
$e_{G/N} = N$. So, suppose some element $Na$ of $G/N$ belonged to $\ker(\psi)$. This means that $\psi(Na) = e_H$.

By definition of $\psi$, this implies $f(a) = e_H$.

Thus $a \in \ker(f) = N$ so $Na = N = e_{G/N}$. Thus:

$Na \in \ker(\psi) \implies Na = e_{G/N} = N$

so $\ker(\psi) = \{N\} = \{e_{G/N}\}$. Thus $\psi \neq$ injective.

Lastly, to show $\psi$ is surjective, suppose $h \in H$ is given. Since $f$ is surjective, we know that $h = f(a)$ for some $a \in G$. Hence

$\psi(Na) = f(a) = h$.

Thus $\psi$ is surjective.
Therefore, we have shown that $\Psi$ is an isomorphism of $G/N$ onto $H$. So, it remains only to show that $f = \Psi \circ \pi$. To show this, suppose $a \in G$. Then:

$$(\Psi \circ \pi)(a) = \Psi(\pi(a)) = \Psi(Na) = f(a).$$

Thus $\Psi \circ \pi = f$. \hfill \Box$$
Remark

What if $f$ isn't surjective? In this case $\text{range}(f)$ is a subgroup of $H$, and

$$f: G \to \text{range}(f)$$

is surjective. Therefore, if $N = \ker(f)$ then by the First Homomorphism Theorem implies that

$$G/N \cong \text{range}(f).$$
Example

Suppose $G = \langle a \rangle$ is a finite cyclic group of order $m$.

Define

$$f: \mathbb{Z} \rightarrow G$$

$$f(k) = a^k$$

Exercise: Show that $f$ is a surjective homomorphism.

The kernel of $f$ is

$$\ker(f) = \{ k \in \mathbb{Z} : a^k = e \}$$

$$= \{ km : k \in \mathbb{Z} \} \quad \text{since } m = o(a)$$

$$= m \mathbb{Z} = [m]$$

The First Homomorphism Theorem implies that

$$G \cong \mathbb{Z}/m \mathbb{Z} = \mathbb{Z}_m.$$  

Thus, every cyclic group of order $m$ is isomorphic to $\mathbb{Z}_m.$
Inverse image: Recall the definition
\[ H = \varphi^{-1}(H') = \{ a \in G : \varphi(a) \in H' \} \]

Suppose \( \varphi : G \to G' \) is a surjective homomorphism with kernel \( K \). If \( H' \) is a subgroup of \( G' \), then its inverse image \( H = \varphi^{-1}(H') \) is a subgroup of \( G \). What kind of subgroup is it?

\[ \begin{array}{c}
G \\
\varphi \\
H \\
\varphi^{-1}(H') \\
H' \\
G' \end{array} \]

Since \( H' \) contains \( e_{G'} \), the inverse image of \( H' \) will contain everything in \( G \) that maps to \( e_{G'} \), which is \( K \). So \( H = \varphi^{-1}(H') \) will contain \( K \).

Further, if we restrict our attention to \( \varphi \) on \( H \), i.e., take \( H \) as the domain of \( \varphi \), then \( \varphi \) is a surjective mapping of \( H \) onto \( H' \), and its kernel is \( K \). Therefore \( K \) is a normal subgroup of \( H \), and the First Homomorphism Theorem therefore implies that

\[ H' \cong H/K. \]
Thus, every subgroup $H'$ of $G'$ is isomorphic to $H/K$ when $H$ is a subgroup of $G$.

**Exercises**

Show that if $H'$ is normal in $G'$ $(H' \triangleleft G')$ then $H$ is normal in $G$ $(H \triangleleft G)$.

**Summary: The Correspondence Theorem**

Suppose $\varphi: G \rightarrow G'$ is a homomorphism of $G$ onto $G'$ and let $K = \ker(\varphi)$. If $H'$ is a subgroup of $G'$, then its inverse image

$$H = \varphi^{-1}(H') = \{a \in G : \varphi(a) \in H'\}$$

is a subgroup of $G$, $H \triangleleft K$, & $H/K \cong H'$. If $H' \triangleleft G'$, then also $H \triangleleft G$. 

Motivation for the Second Homomorphism Theorem

Let \( H = \text{x-y plane in } \mathbb{R}^3 \) (subgroup under +)
\( K = \text{y-z plane in } \mathbb{R}^3 \)

Then \( H + K = \mathbb{R}^3 \) (generic group notation would be \( HK \))

and \( H + K / H \cong \text{z-axis} \).

Also, \( HK = \text{y-axis} \)

and \( K / HK \cong \text{z-axis} \).

Thus \( H + K / H \cong K / HK \).

Note that in this example, all the subgroups are normal since \( \mathbb{R}^3 \) is abelian.
Second Homomorphism Theorem

Let $H$ be a subgroup of $G$ & $N$ a normal subgroup of $G$. Then:

a. $HN = \{hn : h \in H, n \in N\}$ is a subgroup of $G$.

b. $HN/N$ is a normal subgroup of $H$.

c. $HN/N \cong H/\text{HN}$

Proof: Exercise.

Hint: Define $f : H \rightarrow HN/N$

\[ f(h) = hN \]

Show that $f$ is a surjective homomorphism of $H$ onto $HN/N$ and that $\ker(f) = \text{HN}$. Then apply the First Homomorphism Theorem.
Third Homomorphism Theorem

Let \( \varphi : G \to G' \) be a homomorphism of \( G \) onto \( G' \), & set \( K = \ker(\varphi) \).

Suppose \( N' \triangleleft G' \), & set \( N = \varphi^{-1}(N') \). Then

\[
G/N \cong G'/N'. \quad (*)
\]

Note: By the First Homomorphism Theorem,

\[
G' \cong G/K \quad \text{and} \quad N' \cong N/K
\]

so we can reword (*) as

\[
G/N \cong G/K \cap N/K.
\]
Proof:

Define \[ f: G \rightarrow G'/N' \]
\[ f(a) = N'\varphi(a) \]

Claim: \( f \) is a surjective homomorphism.

First, to show it is onto, choose any coset \( N'b \in G'/N' \), i.e., \( b \) is any element of \( G' \).

Since \( \varphi \) is onto, \( b = \varphi(a) \) for some \( a \in G \). Hence \( N'b = N'\varphi(a) = f(a) \). Thus \( f \) is onto.

Now to show \( f \) is a homomorphism, suppose that \( a, b \in G \). Then

\[ f(ab) = N'\varphi(ab) \]
\[ = N'\varphi(a)\varphi(b) \text{ since } \varphi \text{ is a homomorphism} \]
\[ = (N'\varphi(a))(N'\varphi(b)) \text{ since } N' \text{ is normal} \]
\[ = f(a)f(b). \]

Thus \( f \) is a homomorphism.
Next we claim that \( \ker(f) = N \). To see this, suppose that \( a \in \ker(f) \). Then \( f(a) = N' \), the identity element of \( G'/N' \). Hence \( N' \psi(a) = f(a) = N' \), so \( \psi(a) \in N' \). Therefore \( a \in \psi^{-1}(N') = N \).

Thus \( \ker(f) \subseteq N \).

On the other hand, if \( a \in N = \psi^{-1}(N') \) then \( \psi(a) \in N' \), so \( f(a) = N' \psi(a) = N' \). Thus \( a \in \ker(f) \), so \( N \subseteq \ker(f) \).

Thus we have shown that \( f \) is a surjective homomorphism of \( G \) onto \( G'/N' \) with kernel \( N \).

The First Homomorphism Theorem therefore implies that \( G/N \cong G'/N' \).

Exercise: What exactly is \( \psi \), the isomorphism \( \psi \) from \( G/N \) onto \( G'/N' \)? Find a formula for it (in terms of \( \psi \)).