2.9 Direct Products

If $S, T$ are two arbitrary sets, then their Cartesian product is the set of all ordered pairs of element of $S$ with elements of $T$:

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$ 

If $S, T$ are finite, then $|S \times T| = |S| \cdot |T|.$

When $G, H$ are groups, their Cartesian product also has a natural group structure.

**Exercise**

Let $G, H$ be groups. Define an operation on $G \times H$ by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2).$$

Show that $G \times H$ is a group under this operation with identity element

$$e_{G \times H} = (e_G, e_H)$$

and inverses

$$(a, b)^{-1} = (a^{-1}, b^{-1}).$$
Example \( \mathbb{Z}_3 \times \mathbb{Z}_2 \) (with \( \mathbb{Z}_3, \mathbb{Z}_2 \) under addition)

As a set, \( \mathbb{Z}_3 \times \mathbb{Z}_2 \) has 6 elements:

\[
\mathbb{Z}_3 \times \mathbb{Z}_2 = \{ (0,0), (1,0), (2,0), (0,1), (1,1), (2,1) \}
\]

Since the operation in both \( \mathbb{Z}_3, \mathbb{Z}_2 \) is addition, "powers" of an element mean repeated additions. For example, consider the "powers" (multiples) of \( (1,1) \):

\[
0 \cdot (1,1) = (0,0)
\]

\[
1 \cdot (1,1) = (1,1)
\]

\[
2 \cdot (1,1) = (1,1) + (1,1) = (2,0)
\]

\[
3 \cdot (1,1) = (1,1) + (1,1) + (1,1) = (0,1)
\]

\[
4 \cdot (1,1) = \cdots = (1,0)
\]

\[
5 \cdot (1,1) = \cdots = (2,1)
\]

\[
6 \cdot (1,1) = \cdots = (0,0)
\]

Hence \( (1,1) \) has order 6, so \( \mathbb{Z}_3 \times \mathbb{Z}_2 \) is cyclic:

\[
\mathbb{Z}_3 \times \mathbb{Z}_2 = \langle (1,1) \rangle.
\]
Thus \( \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \).

Exercise: Exhibit an isomorphism between \( \mathbb{Z}_3 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_6 \) explicitly.

Exercise
Show that \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) has order 4 but is NOT cyclic, so \( \mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \).

Exercise
Show that \( \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) are the "only" groups of order 4, in the sense that if \( |G| = 4 \), then either \( G \cong \mathbb{Z}_4 \) or \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Exercise
Extend the definition of product to finitely many groups. That is, if \( G_1, \ldots, G_n \) are groups, show that
\[
G_1 \times \cdots \times G_n = \{(a_1, \ldots, a_n) : a_i \in G_i, \ldots, a_n \in G_n\}
\]
is a group under an appropriate operation. This group is called \( \Delta \) (external) direct product of \( G_1, \ldots, G_n \).

**Exercise**

Let \( G, H \) be groups. Show that \( G \times H \) has a subgroup isomorphic to \( G \), & a subgroup isomorphic to \( H \), namely,

\[
G^* = \{ (g, e_H) : g \in G \}
\]

\[
H^* = \{ (e_G, h) : h \in H \}
\]

**Illustration**

Diagram of \( G \times H \) with points labeled: \( (e_G, h) \), \( (g, h) \), \( (e_G, e_H) \), \( (g, e_H) \), \( G^* \), \( H^* \).
Show further that:

a. \( G^* \cap H^* = \{ e_{G \times H} \} = \{ (e_G, e_H) \} \)

b. \( G^*, H^* \) are each normal in \( G \times H \).

Let us show explicitly one more important fact:

c. \( G \times H = G^* \cdot H^* \)

Proof:

By definition,

\[
G^* \cdot H^* = \{ xy : x \in G^*, y \in H^* \}
\]

\[
= \{ (g, e_H)(e_G, h) : g \in G, h \in H \}
\]

\[
= \{ (ge_G, e_Hh) : ge_G, he_H \}
\]

\[
= \{ (g, h) : ge_G, he_H \}
\]

\[
= G \times H.
\]
This is an important fact: $G \times H$ has normal subgroups $G^*$, $H^*$ which satisfy

a. $G^* \cap H^* = \{e_{G \times H}\}$

b. $G^* H^* = G \times H$.

Conversely, we'll show now that whenever a group has normal subgroups that satisfy analogous conditions, then that group is isomorphic to a direct product.

**Theorem**

If a group $G$ has normal subgroups $M$, $N$ s.t.

a. $M \cap N = \{e\}$

b. $MN = G$

then $G \cong M \times N$. 
Proof:

Define

\[ F : M \times N \rightarrow G \]

\[ F((m,n)) = mn \]

\[ M \times N \]
set of ordered pairs \((m,n)\)

\[ N^* \]

\[ M^* \]

\[ G \]
set of elements

\[ MN = \text{subset of } G \]
consisting of elements \(mn\) with \(m \in M, n \in N\)

Homomorphism

Suppose \((m,n) \) & \((k,l)\) belong to \(M \times N\).

Then:
\[ F((m,n)(k,l)) = F((mk, nl)) = (mk)(nl) \]

while

\[ F((m, n)) F((k, l)) = (mn)(kl). \]

So, we need to show that \( mknl = mnkl \), which means that we must show that \( kn = nk \).

Now \( k \in M \) & \( n \in N \), & both subgroups are normal.

Here

\[ nk \, n^{-1} \, k^{-1} = (n \, k \, n^{-1}) \, k^{-1} \in M \]

and

\[ n \, k \, n^{-1} \, k^{-1} = n \, (k \, n^{-1} \, k^{-1}) \in N. \]

Thus \( nk \, n^{-1} \, k^{-1} \in MN \cap N = \{ e \} \). Hence

\[ nk \, n^{-1} \, k^{-1} = e, \text{ so } nk = kn, \text{ and} \]

Therefore \( F \) is a homomorphism.
Injective

Suppose \( F((m,n)) = e \). Then \( mn = e \). Hence

\[ m = n^{-1} \in \mathbb{N} \quad \text{and} \quad m \in M \]

which implies \( m \in M \cap N = \{ e \} \). Hence \( m = n = e \).

Thus \( (m,n) = (e,e) = e_{M \times N} \), so \( F \) is injective.

Surjective

If \( x \in G \) then since \( G = M \times N \) we have \( x = mn \) for some \( m \in M \) and \( n \in N \). Therefore

\[ F((m,n)) = mn = x, \quad \text{so} \quad F \text{ is surjective.} \]

Thus \( F \) is an isomorphism, so \( G \cong M \times N \).
Exercise

Let \( G = \{e, r, r^2, r^3, a, b, c, d\} \) be the dihedral group of order 8. Let

\[ M = \langle r \rangle = \{e, r, r^2, r^3\}, \quad N = \langle a \rangle = \{e, a\}. \]

Show that

\[ G = MN \text{ and } M \cap N = \{e\}. \]

Prove, however, that \( G \neq M \times N \).

Why does this not contradict the preceding theorem?

Exercise

Show that if \( G \) & \( H \) are both abelian, then \( G \times H \) is abelian.

Exercise:

Prove or find a counterexample: if \( G \) & \( H \) are cyclic groups then \( G \times H \) is cyclic.
Definition

If $M, N$ are normal subgroups of a group $G$ then we say that $G$ is the (internal) direct product of $M \& N$ if

$$G = MN \quad \text{and} \quad MNN = \{e\}.$$  

Note that, by the preceding theorem, if $G$ is the internal direct product of $M \& N$ then $G$ is isomorphic to the external direct product $M \times N$.

Exercise

Show that if $G$ is the internal direct product of normal subgroups $M, N$, then

$$x \in G \implies \text{unique } m \in M, n \in N \text{ s.t. } x = mn.$$
2.10 Finite Abelian Groups

We will not prove this result, but the finite abelian groups can be completely characterized.

Theorem
Every finite group is a direct product of cyclic subgroups.

Further, each cyclic subgroup can be taken to have prime-power order.

Corollary
If $|G| = p^n$, then

$$G \cong \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$$

where $n_1 + \cdots + n_k = n$.

Example
An abelian group of order 27 is isomorphic to one of

$$\mathbb{Z}_{27}, \quad \mathbb{Z}_9 \times \mathbb{Z}_3, \quad \text{or} \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

None of these three groups is isomorphic to the others.
Example

An abelian group of order 20 is isomorphic to one of

\[ \mathbb{Z}_4 \times \mathbb{Z}_5 \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \]

\[ \cong \mathbb{Z}_{20} \text{ (cyclic)} \quad \text{not cyclic} \]