4.4 Maximal Ideals

**Definition**
A ring $R$ is *simple* if it has no nontrivial ideals, i.e., \{0\} & $R$ are the only ideals.

**Definition**
An ideal $M$ in a ring $R$ is a **maximal ideal** if

a. $M \subsetneq R$

b. There are no ideals $I$ with $M \subsetneq I \subsetneq R$.

**Note:** Thus

$R$ is a simple ring $\iff$ \{0\} is a maximal ideal.
Theorem
Let $M$ be a proper ideal in a ring $R$. Then:

\[ M \text{ is maximal } \iff R/M \text{ is simple. } \]

Proof:

\[ \Leftarrow \]
Suppose that $R/M$ is simple. Suppose that $I$ is an ideal in $R$ & $M \subseteq I \subseteq R$.
Consider the canonical map

\[ \pi: R \to R/M \]

\[ \pi(a) = a + M. \]
Consider

\[ J = \pi(I) = \{ a + M : a \in I \} \]

Exercise: Show $J$ is an ideal in $R/M$.

But $R/M$ is simple, so either $J = \{M\}$ (since $M$ is the zero element of $R/M$), or $J = R/M$. 

\[ \text{[Handwritten annotations]} \]
Case 1: \( J = \{ M \} \).

We claim that this implies that \( I = M \).

To see this, suppose that \( a \in I \). Then, by definition of \( J \), we have \( a + M \in J \). But \( J = \{ M \} \), so this implies \( a + M = M \). Hence \( a \in M \), so we have shown that \( I \subseteq M \).

Exercise: Show that \( M \subseteq I \).

Thus, in this case we have \( I = M \).

Case 2: \( J = R/M \).

We claim that this implies that \( I = R \).

Note that we have \( I \subseteq R \), so we only have to show the reverse inclusion. Suppose that \( a \in R \). Then \( a + M \in R/M = J \), so \( a + M = b + M \) for some \( b \in I \). But then \( a - b \in M \), and
$M \subseteq I$, so $a-b \in I$. Since $b \in I$, we conclude $a = (a-b) + b \in I$. Therefore $R = I$.

Thus there are only two possibilities for $I$: either $I = M$ or $I = R$. Therefore $M$ is maximal.

$\Rightarrow$ Exercise.

Hint: Suppose that $J$ is an ideal in $R/M$. Show that

$$ I = \pi^{-1}(J) = \{a \in R : a+M \in J \} $$

is an ideal in $R$, and $M \subseteq I \subseteq R$. Since $M$ is maximal, this leaves only two possibilities.
Corollary

Let $R$ be a commutative ring with identity, and let $M$ be a proper ideal in $R$. Then:

$M$ is maximal $\iff R/M$ is a field.

Proof:

$\Leftarrow$ Suppose that $R/M$ is a field. Then we know by earlier results that $R/M$ is simple. The preceding theorem therefore implies that $M$ is maximal.

$\Rightarrow$ Suppose that $M$ is maximal. Then $R/M$ is a commutative ring with 1 that has no nontrivial ideals. We proved earlier that this implies that $R/M$ is a field.
Example/Exercise

Consider \( R = \mathbb{Z} \). Suppose \( \text{det } n > 0 \) is composite, i.e., \( n = kl \) with \( k, l > 1 \).

Exercise: Show \( n \mathbb{Z} \not\subseteq k \mathbb{Z} \cap \mathbb{Z} \).

Thus \( n \mathbb{Z} \) is not a maximal ideal. Therefore \( \mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z} \) is not a field.

Suppose \( p > 0 \) is prime. Show that \( p \mathbb{Z} \) is a maximal ideal, & conclude that \( \mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \) is a field.