Here are a few practice problems on groups. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical to mine—there are often many ways to solve a problem—but it does need to be CORRECT.

1. Suppose that \( N \) is a normal subgroup of a group \( G \). Show that if \( a \in G \) has finite order \( o(a) \), then \( Na \) in \( G/N \) has finite order \( m \) with \( m | o(a) \).

**Solution**

Let \( n = o(a) \). Then \( a^n = e \), so

\[
(Na)^n = Na^n = Ne = N,
\]

which is the identity element of \( G/N \). Therefore \( Na \) has finite order, and its order must divide \( n \). In other words, we must have \( o(Na) | n \). □

**Question:** Must it be true that \( o(Na) = o(a) \)? (No. Try an example!)

2. Suppose that \( H \) and \( K \) are groups. I won’t prove it here, but you should be able to prove that the Cartesian product

\[
H \times K = \{(h, k) : h \in H, k \in K\}
\]

is a group with group operation \((h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)\). Prove the following facts about this group.

(a) Show that \( H^* = \{(h, e_K) : h \in H\} \) and \( K^* = \{(e_H, k) : k \in K\} \) are normal subgroups of \( H \times K \).

**Solution**

The fact that \( H^* \) and \( K^* \) are subgroups of \( H \times K \) is a straightforward verification of the definition of subgroup, and I will leave this part to you. To see that \( H^* \) is a normal subgroup, suppose that \( (h, e_K) \in H^* \) and \( (x, y) \) is any element of \( H \times K \). Then:

\[
(x, y)(h, e_K)(x, y)^{-1} = (x, y)(h, e_K)(x^{-1}, y^{-1}) = (xhx^{-1}, ye_Ky^{-1}) = (xhx^{-1}, e_K) \in H^*.
\]

This shows that \( (x, y)H^*(x, y)^{-1} \subseteq H^* \), so we conclude that \( H^* \) is normal. A completely symmetric proof then shows that \( K^* \) is normal as well. □

(b) Show that \( H \cong H^* \) and \( K \cong K^* \).

**Solution**

To see that \( H \) and \( H^* \) are isomorphic, define a mapping \( f : H \rightarrow H^* \) by

\[
f(h) = (h, e_K), \quad h \in H.
\]

This is a homomorphism because given any \( a, b \in H \) we have

\[
f(ab) = (ab, e_K) = (a, e_K)(b, e_K) = f(a)f(b).
\]
To complete the proof, you must verify that $f$ is a bijection (do it!). Then you have shown that $f$ is an isomorphism, so you conclude that $H \cong H^*$. A similar argument shows that $K \cong K^*$. □

(c) Prove that $(H \times K)/K^* \cong H$. Hint: Find a function that maps $H \times K$ to $H$, and then use the First Homomorphism Theorem.

Solution

Define $f : H \times K \to H$ by the rule

$$f(h, k) = h, \quad (h, k) \in H \times K.$$

This is a homomorphism because

$$f((h_1, k_1)(h_2, k_2)) = f(h_1h_2, k_1k_2) = h_1h_2 = f(h_1, k_1)f(h_2, k_2).$$

It is certainly surjective because if $h \in H$ then $f(h, e_K) = h$. We claim that the kernel of $f$ is $K^*$. First, if $(e_H, k) \in K^*$ then $f(e_H, k) = e_H$, so $(e_H, k) \in \ker(f)$. Second, if $(h, k) \in \ker(f)$ then $h = f(h, k) = e_H$, so $(h, k) = (e_H, k) \in K^*$. Thus $\ker(f) = K^*$.

Thus, we have a surjective homomorphism $f$ that maps $H \times K$ onto $H$ and has kernel $K^*$. The First Homomorphism Theorem therefore tells us that $H \cong (H \times K)/K^*$. □