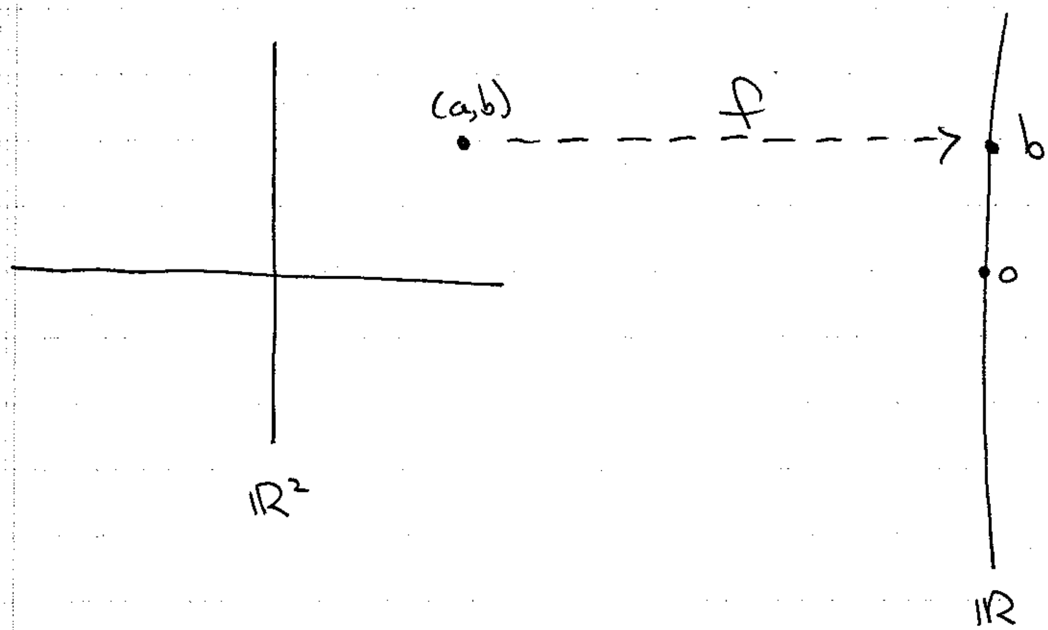
2.7 The Homomorphism Theorems

Motivation

Consider the map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$f((a,b)) = b$$

f sends the point $(a,b) \in \mathbb{R}$ to the number $b \in \mathbb{R}$.



If we think of $\mathbb{R}^2$ & $\mathbb{R}$ as being groups under addition, then f is a homomorphism because

$$f\big((a_1,b_1)+(a_2,b_2)\big) = f\big((a_1+a_2, b_1+b_2)\big)$$
$$= b_1 + b_2$$
$$= f((a_1,b_1)) + f((a_2,b_2)).$$

f is surjective, but it is not injective — any two points with the same second coordinate are mapped to the same place. In fact,

$$N = \ker(f) = \{(x,0) : x \in \mathbb{R}\} = x\text{-axis in } \mathbb{R}^2.$$

Recall that we earlier used $\mathbb{R}^2/N$ as an example motivating the definition of the quotient group. The cosets of N are

$$N + (a,b) = N + (0,b) = \{(x,b) : x \in \mathbb{R}\} = L_b,$$

the line a height b. The quotient group $\mathbb{R}^2/N$ is the set of all cosets of N:

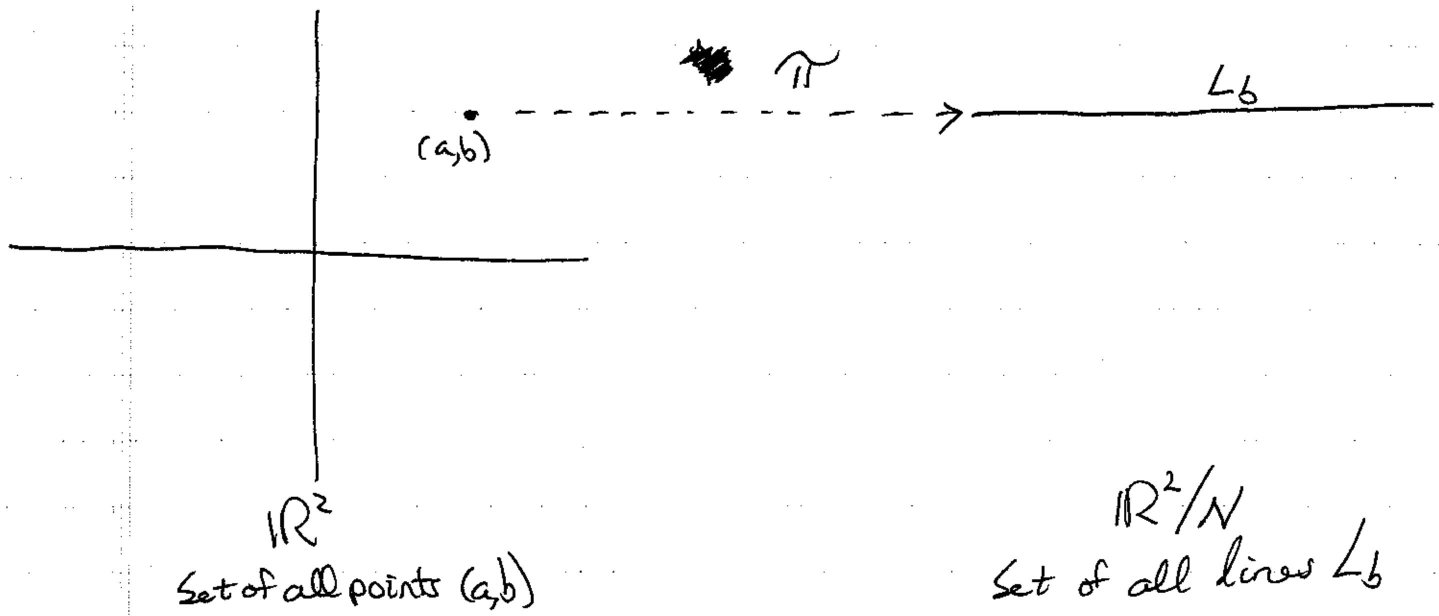
$$\mathbb{R}^2/N = \{L_b : b \in \mathbb{R}\},$$

the set of all horizontal lines in $\mathbb{R}^2$.

The canonical projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/N$ sends a point (a,b) to the coset $N + (a,b) = L_b$:

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/N$$

$$\pi((a,b)) = L_b$$



π is a homomorphism – we saw that the operation in $\mathbb{R}^2/N$ is given by $L_b + L_c = L_{b+c}$, so

$$\pi((a_1,b_1) + (a_2,b_2)) = \pi((a_1+a_2, b_1+b_2))$$

$$= L_{b_1+b_2}$$

$$= L_{b_1} + L_{b_2}$$

$$= \pi((a_1,b_1)) + \pi((a_2,b_2)).$$

So now we have two maps on $\mathbb{R}^2$:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{R}^2 \arrow[r, "f", "\substack{\text{homomorphism,}\\ \text{surjective}}"'] \arrow[d, "\pi"', "\substack{\text{homomorphism,}\\ \text{surjective}}"] & \mathbb{R} \\
\mathbb{R}^2/N &
\end{tikzcd}$$

These two maps are related by the fact that

$N = \ker(f)$ & π maps $\mathbb{R}^2$ onto $\mathbb{R}^2/N$.

The kernel <u>of f</u> is used to make the quotient group $\mathbb{R}^2/N$.

Now compare the two ranges $\mathbb{R}$ & $\mathbb{R}^2/N$.

They are actually very similar!

$\mathbb{R}$ = set of numbers

$= \{b : b \in \mathbb{R}\}$

Operation is

$b + c$ = usual sum of b & c

$\mathbb{R}^2/N$ = set of lines

$= \{L_b : b \in \mathbb{R}\}$

Operation is

$L_b + L_c = L_{b+c}$.

These groups have the <u>same structure</u>, and they are actually <u>isomorphic</u>. That is, we claim that

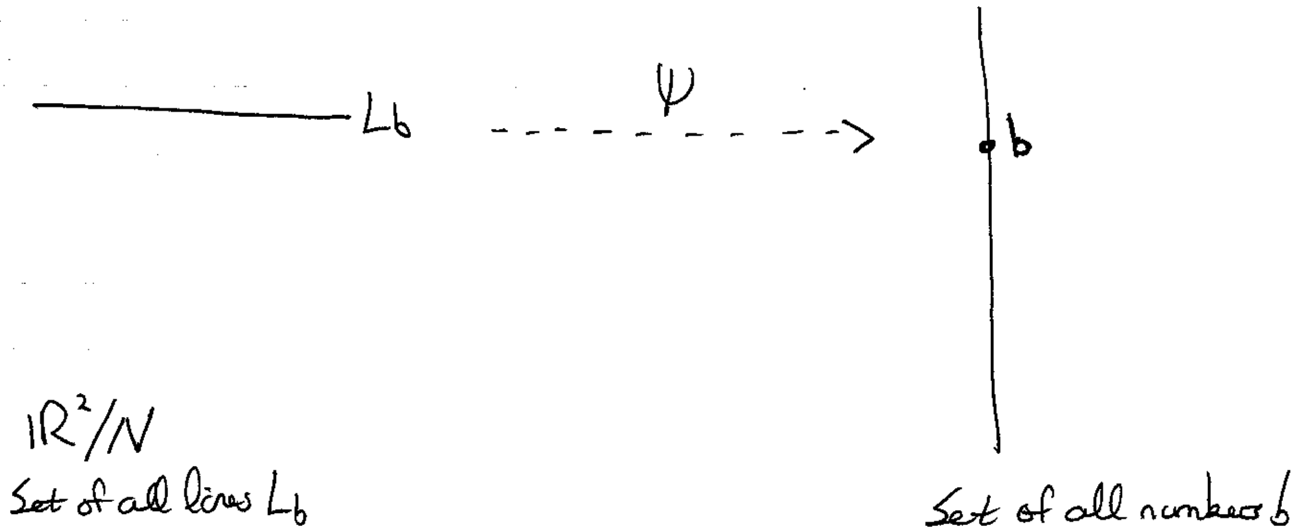
$$\mathbb{R}^2/N \cong \mathbb{R}.$$

To prove this, we must show that there exists an isomorphism ψ that maps $\mathbb{R}^2/N$ to $\mathbb{R}$.

Define

$$\psi : \mathbb{R}^2/N \longrightarrow \mathbb{R}$$
$$\psi(L_b) = b$$

ψ maps the line at height b to the number b.



ψ is a bijection, & it is a homomorphism because

$$\begin{aligned}\psi(L_b + L_c) &= \psi(L_{b+c}) \\ &= b + c \\ &= \psi(L_b) + \psi(L_c)\end{aligned}$$

This is a general phenomenon!!

$$\begin{tikzcd} G \arrow[r, "f"', "\text{surjective homomorphism}"] \arrow[d, "\pi\ \text{canonical projection}"'] & H \\ G/N,\ N = \ker(f) \arrow[ur, "\text{Then } \exists \text{ isomorphism } \psi"'] & \end{tikzcd}$$

The following result is known by many names, including

First Homomorphism Theorem
First Isomorphism Theorem
The Homomorphism Theorem
The Isomorphism Theorem

<u>First Homomorphism Theorem</u>

Let $f : G \to H$ be a <u>surjective</u> homomorphism of G onto H, & let $N = \ker(f)$. Then

$$G/N \cong H$$

The isomorphism is

$$\Psi : G/N \longrightarrow H$$
$$\Psi(Na) = f(a)$$

Further,

$$f = \Psi \circ \pi$$

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\
G/N \arrow[ur, dashed, "\exists \text{ isomorphism } \Psi"'] &
\end{tikzcd}$$

Proof,

We have a proposed definition of the isomorphism Ψ from G/N to H, namely, $\Psi(Na) = f(a)$. But does this definition make any sense? We could have $Na = Nb$ even though $a \neq b$. In this case $\Psi(Na)$ & $\Psi(Nb)$ must have the same definition, but should they be defined to be $f(a)$ or $f(b)$? The only way that this definition will be OK is if

$$Na = Nb \implies f(a) = f(b).$$

But we don't know that this is true, we have to prove it. To do this, suppose that we had $Na = Nb$. Then $ab^{-1} \in N = \ker(f)$.

Therefore

$$e_H = f(ab^{-1}) = f(a)\, f(b)^{-1}.$$

because $ab^{-1} \in \ker(f)$

because f is a homomorphism.

Hence we do indeed have $f(a) = f(b)$. Therefore the proposed function Ψ is well-defined, ie., it makes sense.

Now we have to show that Ψ actually is an isomorphism. First let's show that it is a homomorphism. Given $Na, Nb \in G/N$, we have

$$
\begin{aligned}
\Psi((Na)(Nb)) &= \Psi(N(ab)) && \text{def. of group op in } G/N \\
&= f(ab) && \text{def of } f \\
&= f(a)f(b) && \text{since } f \text{ is a homomorphism} \\
&= \Psi(Na)\Psi(Nb) && \text{def of } f.
\end{aligned}
$$

Thus Ψ is indeed a homomorphism.

Since Ψ is a homomorphism, to show it is injective, we just have to show that $\mathrm{Ker}(\Psi)$ contains only the identity element of G/N, which is

$e_{G/N} = N$. So, suppose some element Na of G/N belonged to $\ker(\Psi)$. This means that

$$\Psi(Na) = e_H.$$

By definition of Ψ, this implies

$$f(a) = e_H$$

Thus $a \in \ker(f) = N$, so $Na = N = e_{G/N}$. Thus:

$$Na \in \ker(\Psi) \implies Na = e_{G/N} = N$$

so $\ker(\Psi) = \{N\} = \{e_{G/N}\}$. Thus Ψ is injective.

Lastly, to show Ψ is surjective, suppose $h \in H$ is given. Since f is surjective, we know that $h = f(a)$ for some $a \in G$. Hence

$$\Psi(Na) = f(a) = h.$$

Thus Ψ is surjective.

Therefore, we have shown that ψ is an isomorphism of G/N onto H. So, it remains only to show that $f = \psi \circ \pi$. To show this, suppose $a \in G$. Then

$$
\begin{aligned}
(\psi \circ \pi)(a) &= \psi(\pi(a)) \\
&= \psi(Na) \\
&= f(a).
\end{aligned}
$$

Thus $\psi \circ \pi = f$. $\blacksquare$

Remark

What if f isn't surjective? In this case $\mathrm{range}(f)$ is ~~a~~ a subgroup of H, and

$$f : G \longrightarrow \mathrm{range}(f)$$

is surjective. Therefore, if $N = \ker(f)$ then the First Homomorphism Theorem implies that

$$G/N \cong \mathrm{range}(f).$$

Example

Suppose $G = \langle a \rangle$ is a finite cyclic group of order m.
Define

$$f : \mathbb{Z} \longrightarrow G$$
$$f(k) = a^k$$

Exercise: Show that f is a surjective homomorphism.

The kernel of f is

$$\begin{aligned} \ker(f) &= \{k \in \mathbb{Z} : a^k = e\} \\ &= \{km : k \in \mathbb{Z}\} \qquad \text{since } m = o(a) \\ &= m\mathbb{Z} = [m] \end{aligned}$$

The First Homomorphism Theorem implies that

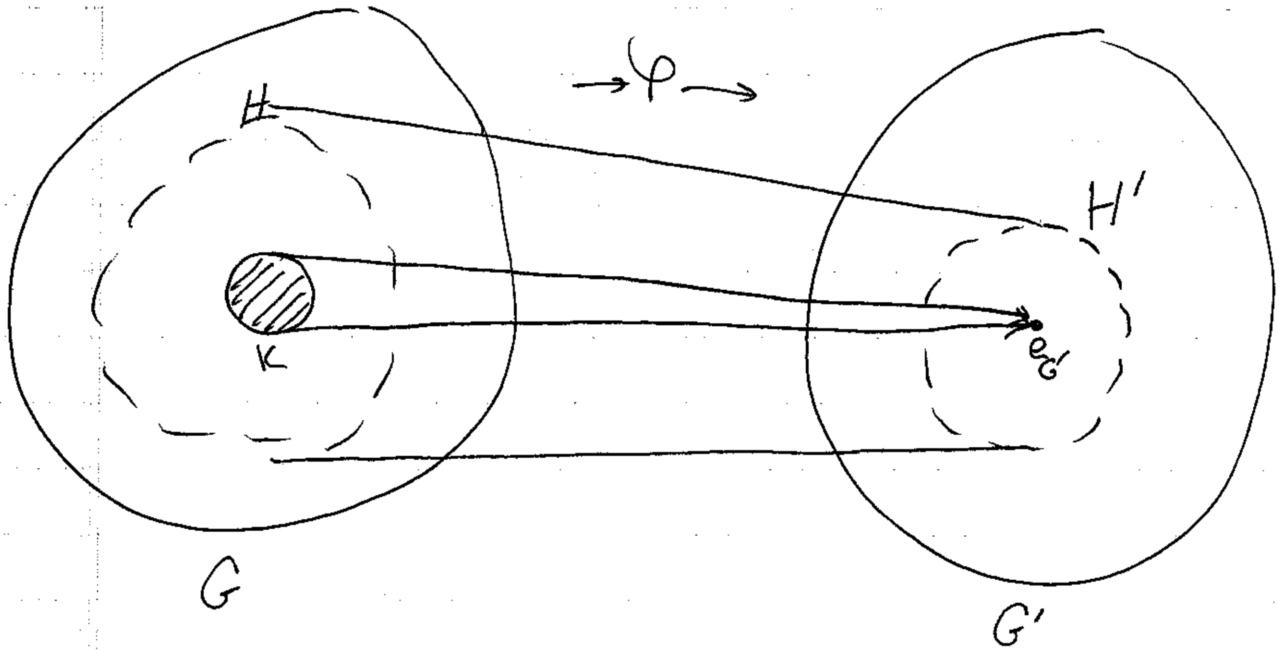
$$G \cong \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m.$$

Thus, <u>every cyclic group of order m is isomorphic to $\mathbb{Z}_m$.</u>

Inverse image: Recall the definition

$$H = \varphi^{-1}(H') = \{a \in G : \varphi(a) \in H'\}$$

Suppose $\varphi : G \to G'$ is ~~a~~ surjective homomorphism with kernel K. If H' is a subgroup of G', then its inverse image $H = \varphi^{-1}(H')$ is a subgroup of G. What kind of subgroup is it?



Since H' contains $e_{G'}$, the inverse image of H' will contain everything in G that maps to $e_{G'}$, which is K. So $H = \varphi^{-1}(H')$ will contain K.

Further, if we restrict our attention to φ on H, i.e., take H as the domain of φ, then φ is a surjective mapping of H onto H', & its kernel is K. Therefore K is a normal subgroup of H, and the First Homomorphism Theorem therefore implies that

$$H' \cong H/K.$$

Thus, every subgroup H' of G' is isomorphic to H/K where H is a subgroup of G.

<u>Exercise:</u>

Show that if H' is normal in G' $(H' \triangleleft G')$ then H is normal in G $(H \triangleleft G)$.

<u>Summary:</u> <u>The Correspondence Theorem</u>

Suppose $\varphi: G \longrightarrow G'$ is a homomorphism of G <u>onto</u> G', & let $K = \ker(\varphi)$. If H' is a subgroup of G', then its inverse image

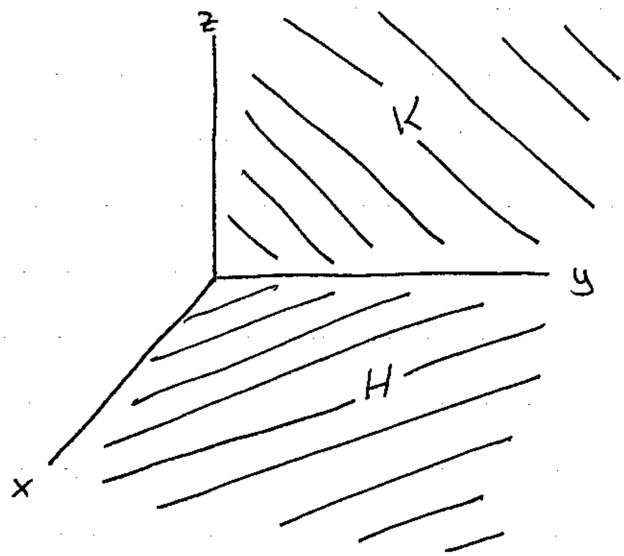
$$H = \varphi^{-1}(H') = \{a \in G : \varphi(a) \in H'\}$$

is a subgroup of G, $H \supseteq K$, & $H/K \cong H'$.

If $H' \triangleleft G'$, then also $H \triangleleft G$.

Motivation for the Second Homomorphism Theorem

Let H = x-y plane in $\mathbb{R}^3$ (subgroup under +)
K = y-z plane in $\mathbb{R}^3$ " " "



Then $H+K = \mathbb{R}^3$ (generic group notation would be HK)

and $H+K/H \cong$ z-axis.

Also, $H \cap K$ = y-axis

and $K/H\cap K \cong$ z-axis.

Thus
$$H+K/H \cong K/H\cap K$$

Note that in this example all the subgroups are normal since $\mathbb{R}^3$ is abelian.

Second Homomorphism Theorem

Let H be a subgroup of G & N a normal subgroup of G. Then:

a. $HN = \{hn : h \in H, n \in N\}$ is a subgroup of G,

b. $H \cap N$ is a normal subgroup of H,

c. $HN/N \cong H/H\cap N$

Proof: Exercise.

Hint: Define

$$f : H \longrightarrow HN/N$$
$$f(h) = hN$$

Show that f is a surjective homomorphism of H onto HN/N and that $\ker(f) = H \cap N$. Then apply the First Homomorphism Theorem.

Third Homomorphism Theorem

Let $\varphi: G \to G'$ be a homomorphism of G onto G', & set $K = \ker(\varphi)$.

Suppose $N' \triangleleft G'$, & set $N = \varphi^{-1}(N')$. Then

$$G/N \cong G'/N'. \quad (*)$$

Note: By the First Homomorphism Theorem,

$$G' \cong G/K \quad \text{and} \quad N' \cong N/K$$

so we can reword $(*)$ as

$$G/N \cong \frac{G/K}{N/K}.$$