

Aside: Functions whose domain is a set of functions.

Let

$$F = \{f : f \text{ is a function from } \mathbb{R} \text{ to } \mathbb{R}\}$$

The set F ~~consists~~ consists of all possible functions that map real numbers to real numbers.

So for example, if $f(x) = e^x$ then $f \in F$.

Notation: Do NOT write $f(x) \in F$.

$f(x)$ denotes the value of f at x , and is a real number. For example, if $f(x) = e^x$ then

$f(1) = e^1 = e$. The number e is not a

function: $f(1) = e \notin F$. The function f

whose rule is $f(x) = e^x$ does belong to F : $f \in F$.

Sometimes (often) we abuse notation and

write $e^x \in F$, but you must realize that

this is incorrect notation, used only because we are lazy.

Now I want to consider the group

$$A(\mathcal{F}) = \{ F: \mathcal{F} \rightarrow \mathcal{F} : F \text{ is a bijection} \}$$

To be an element of $A(\mathcal{F})$, we must have a function $F: \mathcal{F} \rightarrow \mathcal{F}$ that is a bijection.

So, F inputs an element of \mathcal{F} (a function on \mathbb{R}) and outputs a new function ~~new~~

$F(f)$. Since $F(f)$ is a new function mapping

real numbers to real numbers, $F(f)$ is a rule

that maps a number x to a number $F(f)(x)$.

Since the multitude of parentheses is often

~~confusing~~ confusing we sometimes write

$$Ff \quad \text{instead of } F(f).$$

So Ff is a function, & it takes values

$(Ff)(x)$ for $x \in \mathbb{R}$.

Now let us consider some specific examples of functions $F \in A(\mathcal{F})$.

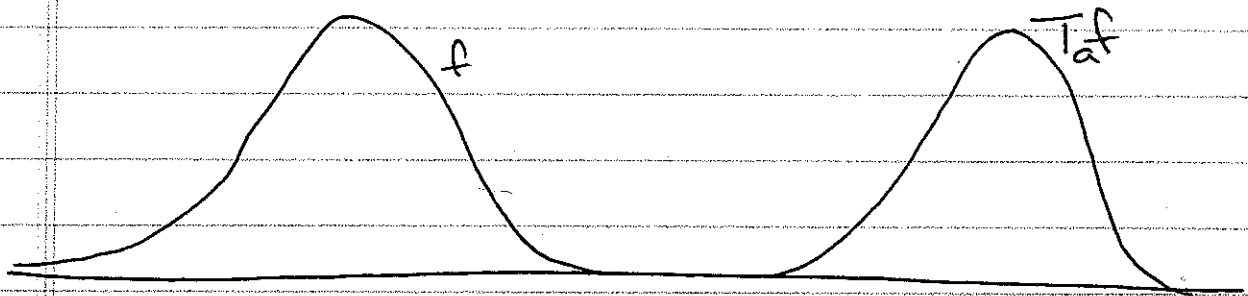
Translation

Given a fixed real number $a \in \mathbb{R}$, we define a function $T_a \in A(\mathcal{F})$ as follows. Given $f \in \mathcal{F}$ we want $T_a f$ to be a new function on \mathbb{R} .

We define $T_a f$ to be the function whose rule is

$$(T_a f)(x) = f(x-a).$$

The graph of the function $T_a f$ is the graph of f shifted a units to the right.



For example, if $a = \pi/2$ and $f(x) = \sin x$, then $T_{\frac{\pi}{2}} f$ is the function whose rule is

$$(T_{\frac{\pi}{2}} f)(x) = f(x - \frac{\pi}{2}) = \sin(x - \frac{\pi}{2}) = \cos x.$$

We have that $T_a: F \rightarrow F$, i.e., T_a maps functions to functions.

Exercise: Show that T_a is a bijection of F onto itself.

This does not mean that $T_a f$ is a bijection for each $f \in F$. Rather what you have to show is that:

T_a is 1-1: if $T_a f = T_a g$ then $f = g$,

T_a is onto: if $g \in F$ then $\exists f \in F$ s.t. $T_a f = g$.

Each different $a \in \mathbb{R}$ gives us a different

$T_a \in A(F)$. However, these are not the only elements of $A(F)$. Let us give another example.

Dilation

Suppose that $a > 0$ is a fixed real number.

Define a function $D_a: \mathcal{F} \rightarrow \mathcal{F}$ as follows.

Given $f \in \mathcal{F}$, we define $D_a f$ to be the function whose rule is

$$(D_a f)(x) = f(ax), \quad x \in \mathbb{R}.$$

The functions $f \in \mathcal{F}$ need not be bijections of \mathbb{R} onto \mathbb{R} . However, the mapping from f to $D_a f$ is a bijection of \mathcal{F} onto \mathcal{F} .

Exercises: Given a fixed $a \in \mathbb{R}$, show that

$D_a \in A(\mathcal{F})$, i.e., D_a is a bijection of \mathcal{F} onto \mathcal{F} .

Example

Suppose ~~that~~ $a = 2$ and $f(x) = e^x$.

Then $D_a f$ is the function whose rule is

$$(D_2 f)(x) = f(2x) = e^{2x}.$$

Compositions

Given ~~any~~ $a > 0$ & $b \in \mathbb{R}$, we have that $D_a, T_b \in A(\mathbb{F})$. Since $A(\mathbb{F})$ is a group under compositions, we know that $D_a T_b = D_a \circ T_b$ belongs to $A(\mathbb{F})$. What is the composition $D_a T_b$? It is a bijection of \mathbb{F} onto \mathbb{F} given by

$$(D_a T_b)(f) = D_a(T_b(f)).$$

Now, each of these are functions that map real numbers to real numbers. f is the input to $D_a T_b$, f is not being composed with anything. Rather, the dilation D_a is being composed with the translation T_b , & the result is evaluated at

the function f . Each of

$$(D_a T_b)(f) \quad \text{and} \quad D_a(T_b(f))$$

are functions on \mathbb{R} . To say that they are

equal functions means that they act the

same on each number $x \in \mathbb{R}$, i.e.,

$$((D_a T_b)(f))(x) = (D_a(T_b(f)))(x).$$

The right-hand side says that the function f is

translated by b , then dilated by a , &

the result is evaluated at x . Let us work

out what the result is. For simplicity of

notation, let's cut down on the parentheses &

write $T_b f$ for $T_b(f)$. Then:

$$\begin{aligned}
((D_a T_b)(f))(x) &= (D_a(T_b f))(x) \\
&= (D_a g)(x) \quad \text{where } g = T_b f \\
&= g(ax) \\
&= (T_b f)(ax) \\
&\del{=} (T_b f)(y) \quad \text{where } y = ax \\
&= f(y-b) \\
&= f(ax-b).
\end{aligned}$$

Exercise

Show that

$$((T_b D_a)(f))(x) = f(ax-ab)$$

Notation

Again for simplicity of notation we often omit some parentheses, and write

$$D_a T_b f \text{ for } (D_a T_b)(f) = D_a(T_b(f)).$$

Thus we have shown that

$$(D_a T_b f)(x) = f(ax - b).$$

Important note

We CANNOT write

$$(D_a T_b f)(x) = D_a(T_b(f(x))) \quad !! \quad (*)$$

First, $*$ doesn't make any sense - $f(x)$ is a scalar, not a function, so $T_b(f(x))$ doesn't make sense

You can't write $T_b(\pi)$ - the scalar π is not in the domain of T_b . T_b is a function whose inputs are functions, not numbers. $(T_b(f))(x)$ makes

sense, & we usually abbreviate it as $(T_b f)(x)$,
but $T_b(f(x))$ does NOT make sense.

Second, when you try to write the incorrect
formula (*) you are trying to think of it as

$$(D_a \circ T_b \circ f)(x).$$

However, f is not being composed with anything!

D_a & T_b are functions whose domain is F
and f belongs to F . f is the input to

the composition $D_a \circ T_b$, it is not being
composed with $D_a \circ T_b$. With all the parentheses
written out, $(D_a T_b f)(x)$ means

$$\left((D_a \circ T_b)(f) \right)(x)$$

$$= (D_a(T_b(f)))(x).$$

Exercises

a. Given $a \in \mathbb{R}$, we have that T_a belongs to the group $A(\mathbb{F})$. What is the inverse of T_a in the group? Show that $T_a^{-1} = T_{-a}$.

b. Given $a > 0$, show that $D_a^{-1} = D_{1/a}$.

c. Given $a > 0$ & $b \in \mathbb{R}$, show that

$$T_b D_a = D_a T_{ab}.$$

To do Q3, you must show that

$$(T_b D_a)(f) = (D_a T_{ab})(f) \quad \forall f \in \mathcal{F}.$$

And to do Q3, you must show that

$$\left((T_b D_a)(f) \right)(x) = \left((D_a T_{ab})(f) \right)(x) \quad \forall f \in \mathcal{F} \\ \forall x \in \mathbb{R}.$$

Usually we write Q3 in shorthand notation as

$$(T_b D_a f)(x) = (D_a T_{ab} f)(x) \quad \forall f \in \mathcal{F}, \forall x \in \mathbb{R}.$$

Exercises

a. Show that $H = \{T_a\}_{a \in \mathbb{R}}$ is a subgroup of $A(F)$.

b. ~~Let~~ Let \mathbb{R} be the group of reals under addition.

Define

$$U: \mathbb{R} \longrightarrow H$$

$$U(a) = T_a$$

Show that U is an isomorphism, i.e., $H \cong \mathbb{R}$.

Exercises

a. Show that $K = \{D_a\}_{a > 0}$ is a subgroup of $A(F)$.

b. Let $\mathbb{R}^+ = (0, \infty)$ be the positive reals under multiplication. Define

$$V: \mathbb{R}^+ \longrightarrow K$$

$$V(a) = D_a$$

Show that V is an isomorphism, so $K \cong \mathbb{R}^+$.

Exercises

a. Show that

$$A = \{D_a T_b : \cancel{a} a > 0, b \in \mathbb{R}\}$$

is a subgroup of $A(\mathbb{F})$.

b. Let G be the affine group (see chapter 1).

Show that $G \cong A$.