

Here are a few practice problems on groups. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

1. Suppose that H is a subgroup of S_n and H contains every 2-cycle $(i_1 i_2)$. Show that $H = S_n$. In other words, we say that *the 2-cycles generate S_n* .

Hint: Show first that H contains every 3-cycle, then every 4-cycle, etc., up to the n -cycles. Then show that H contains every permutation, whether it is a cycle or not.

Solution

Every 3-cycle is a product of 2-cycles, because $(i_1 i_2 i_3) = (i_1 i_3)(i_1 i_2)$. Since H contains every 2-cycle and is closed under composition, it therefore contains every 3-cycle.

Every 4-cycle is a product of a 3-cycle and a 2-cycle, because $(i_1 i_2 i_3 i_4) = (i_1 i_4)(i_1 i_2 i_3)$. Since H contains every 2-cycle and every 3-cycle and is closed under composition, it therefore contains every 4-cycle.

Continuing in this way (more precisely, by doing a proof by induction), we conclude that H contains every n -cycle. Since every permutation can be written as a product of cycles and since H is closed under compositions, we conclude that H contains every permutation. \square

2. Show that $N = \{e, (123), (132)\}$ is a normal subgroup of S_3 .

Solution

We can check this by direct calculation. Using the facts that $(123)^{-1} = (132)$, $(12)^{-1} = (12)$, $(13)^{-1} = (13)$, and $(23)^{-1} = (23)$, we compute that:

$$\begin{aligned} eNe^{-1} &= \{eee^{-1}, e(123)e^{-1}, e(132)e^{-1}\} = \{e, (123), (132)\}, \\ (123)N(123)^{-1} &= \{(123)e(123)^{-1}, (123)(123)(123)^{-1}, (123)(132)(123)^{-1}\} = \{e, (123), (132)\}, \\ (132)N(132)^{-1} &= \{(132)e(132)^{-1}, (132)(123)(132)^{-1}, (132)(132)(132)^{-1}\} = \{e, (132), (123)\}, \\ (12)N(12)^{-1} &= \{(12)e(12)^{-1}, (12)(123)(12)^{-1}, (12)(132)(12)^{-1}\} = \{e, (123), (132)\}, \\ (13)N(13)^{-1} &= \{(13)e(13)^{-1}, (13)(123)(13)^{-1}, (13)(132)(13)^{-1}\} = \{e, (132), (123)\}, \\ (23)N(23)^{-1} &= \{(23)e(23)^{-1}, (23)(123)(23)^{-1}, (23)(132)(23)^{-1}\} = \{e, (123), (132)\}. \end{aligned}$$

Thus for every permutation $\alpha \in S_3$, we have $\alpha N \alpha^{-1} = N$, so N is normal.

Actually, there's a much easier proof: Since N has 3 elements and S_3 has 6, the index of N is 2, and therefore N must be normal. \square

3. Let S_n be the set of all permutations of $\{1, 2, \dots, n\}$. Show that if n is any integer larger than 2, then there exist two functions $f, g \in S_n$ which do not commute.

Solution

Define $f: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ by $f(1) = 2, f(2) = 1$, and $f(k) = k$ for $k = 3, 4, \dots, n$. Define $g: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ by $g(2) = 3, g(3) = 2$, and $f(k) = k$ for $k = 1, 4, 5, \dots, n$. Then f and g are elements of S_n .

We compute that

$$(f \circ g)(1) = f(g(1)) = f(1) = 2 \quad \text{and} \quad (g \circ f)(1) = g(f(1)) = g(2) = 3.$$

Since these are different, the functions $f \circ g$ and $g \circ f$ have different rules, i.e., $f \circ g \neq g \circ f$. \square

4. If α is an r -cycle, must α^k be an r -cycle?

Solution

Sometimes, but not always. For example, if α is an r -cycle and $k = r$ then $\alpha^k = e$, which is a 1-cycle but not an r -cycle. And here is a less trivial example: $(1\ 2\ 3\ 4)^2 = (1\ 3)(2\ 4)$. Thus the square of this 4-cycle is not even a cycle at all, it is a product of two disjoint transpositions. On the other hand, you CAN show that if $r = p$ is PRIME, then α^k is always a cycle. In this case, if p divides k then $\alpha^k = e$, and if p does not divide k then α^k is also a p -cycle. \square

5. Give an example of $\alpha, \beta, \gamma \in S_5$ such that $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, but $\beta\gamma \neq \gamma\beta$.

Solution

Try $\alpha = (1\ 2)$, $\beta = (3\ 4)$, and $\gamma = (3\ 5)$. Then α commutes with β because they are disjoint, and similarly α commutes with γ because they are disjoint, yet $\beta\gamma = (3\ 5\ 4) \neq (3\ 4\ 5) = \gamma\beta$. \square

6. Let α be any permutation in S_n (not necessarily a cycle), and let i be a fixed number between 1 and n . Prove that α moves i if and only if α^{-1} moves i .

Solution

Suppose that $\alpha(i) = j$ where $j \neq i$. Then we know that $\alpha^{-1}(j) = i$. Therefore we can't also have $\alpha^{-1}(i) = i$, because if we did then α^{-1} would not be a bijection. Hence α^{-1} must move i .

The converse direction, showing that if α^{-1} moves i then α moves i , is entirely similar. \square