1. Prove the following statements. Note that the parts of this problem are not related to each other.

   a. Let $K$ be a compact subset of $\mathbb{R}^p$, and let $r > 0$ be fixed. Prove directly from the definition of compact set that there exist finitely many points $x_1, \ldots, x_N \in K$ such that

   \[\forall y \in K, \ \exists n \in \{1, \ldots, N\} \text{ such that } \|y - x_n\| < r.\]

   Hint: Consider balls $B_r(x)$ with $x \in K$.

   b. For each $n \in \mathbb{N}$, define a function $f_n : \mathbb{R} \to \mathbb{R}$ by

   \[f_n(x) = \begin{cases} 1, & n < x < n + 1, \\ 0, & x \leq n \text{ or } x \geq n + 1. \end{cases}\]

   Show that $f_n$ converges pointwise to the zero function on $\mathbb{R}$, but that $f_n$ does not converge uniformly to the zero function.

   c. Let

   \[x_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{n \cdot 2^n}.\]

   Prove directly from the definition that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

   Remark: You can use without proof, if you like, the fact that $\sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m}$.

2. Let $S = \{s_n : n \in \mathbb{N}\}$ be a set of strictly positive real numbers such that $\inf(S) = 0$. Let $A \subset \mathbb{R}^p$. Suppose that $x \in \mathbb{R}^p$ is such that for each $n \in \mathbb{N}$ there exists a point $y_n \in A \setminus \{x\}$ such that $\|x - y_n\| < s_n$. Prove directly from the definition of cluster point that $x$ is a cluster point of $A$.

   Note: The definition of cluster point is as follows: We say that $x$ is a cluster point of $A$ if for every neighborhood $N$ of $x$ there exists a point $y \in A \cap N$ with $y \neq x$.

3. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with $x_n > 0$ for every $n$. Show that if $\lim (x_n^{1/n}) > 1$, then $(x_n)_{n \in \mathbb{N}}$ is not convergent.

   Hint: Show that it is not a bounded sequence.
4. Suppose that for each \( n \in \mathbb{N} \) we are given a piecewise continuous function \( f_n : [0, 1] \to \mathbb{R} \), and another piecewise continuous function \( f : [0, 1] \to \mathbb{R} \). Show that if \( f_n \to f \) uniformly, then 
\[
\sup_{n \in \mathbb{N}} \| f_n \|_\infty < \infty.
\]
Will it also be true that \( \sup \| f_n \|_1 < \infty \)?

5. Let \( A \) be any subset of \( \mathbb{R}^p \). Let \( A^- \) be the closure of \( A \) and let \( \partial A \) be the boundary of \( A \). Prove that
\[
A^- = A \cup \partial A.
\]
Hint: Prove that \( A \cup \partial A \) is closed.