MATH 4317 Real Analysis I

SOME RECOMMENDED PROBLEMS WITH SOLUTIONS

Here are a few practice problems with solutions. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

Problem. Let (x_n) and (y_n) be bounded sequences of nonnegative real numbers. Let $x^* = \limsup\,(x_n)$. Prove that

$$\limsup_{n\to\infty} x_n y_n \;\le\; \Big(\limsup_{n\to\infty} x_n\Big)\Big(\limsup_{n\to\infty} y_n\Big).$$

<u>Solution</u>

Set

$$x^* \;=\; \limsup_{n\to\infty} x_n, \qquad y^* \;=\; \limsup_{n\to\infty} y_n, \qquad z^* \;=\; \limsup_{n\to\infty} x_n y_n.$$

We want to show that $z^* \le x^* y^*$. For $m \in \mathbf{N}$ define

$$u_m \;=\; \sup_{n\ge m} x_n, \qquad v_m \;=\; \sup_{n\ge m} y_n, \qquad w_m \;=\; \sup_{n\ge m} x_n y_n.$$

By one of the equivalent characterizations of limsup, we then have that

$$x^* \;=\; \lim_{m\to\infty} u_m, \quad y^* \;=\; \lim_{m\to\infty} v_m, \quad z^* \;=\; \lim_{m\to\infty} w_m.$$

We have the following inequalities for each m (WHY?):

$$w_m \;\le\; u_m v_m.$$

Hence

$$z^* \;=\; \lim_{m\to\infty} w_m \;\le\; \lim_{m\to\infty} u_m v_m \;=\; \Big(\lim_{m\to\infty} u_m\Big)\Big(\lim_{m\to\infty} v_m\Big) \;=\; x^* y^*.$$

Question: Must we actually have equality, or does there exist an example where equality does not hold? □

Problem. Let (x_n) and (y_n) be bounded sequences of nonnegative real numbers. Let $x^* = \limsup (x_n)$. Prove that if $y^* = \lim (y_n)$ exists, then $\limsup (x_n y_n) = x^* y^*$.

Solution

Set

$$z^* \;=\; \limsup_{n\to\infty} x_n y_n.$$

Choose $\varepsilon > 0$. Then there exists an $N > 0$ such that

$$n > N \implies y^* - \varepsilon < y_n < y^* + \varepsilon.$$

Therefore, if we set

$$u_m \;=\; \sup_{n\geq m} x_n, \qquad v_m \;=\; \sup_{n\geq m} x_n y_n,$$

then for $m > N$ we have

$$v_m \;=\; \sup_{n\geq m} x_n y_n \;\geq\; (y^* - \varepsilon) \sup_{n\geq m} x_n \;=\; (y^* - \varepsilon)\, u_m.$$

Hence

$$z^* \;=\; \lim_{m\to\infty} v_m \;\geq\; (y^* - \varepsilon) \lim_{m\to\infty} u_m \;=\; (y^* - \varepsilon)\, x^*.$$

Since ε is arbitrary, we conclude that $z^* \geq y^* x^*$.

Also, the preceding problem implies that $z^* \leq x^* y^*$ (why?), so we conclude that we have $z^* = x^* y^*$. $\square$