10. Nested Cells & Bolzano-Weierstrass

Definition
A point \( x \in \mathbb{R}^p \) is a cluster point (or accumulation point, or limit point) of \( A \subseteq \mathbb{R}^p \) if

\[
\forall \text{ neighborhood } N \text{ of } x, \exists y \in N \setminus A \text{ s.t. } y \neq x.
\]

That is, to be a cluster point, every neighborhood of \( x \) has to have a point of \( A \) other than \( x \) itself.

This is somewhat similar to, but different from, boundary points.

As with anything involving neighborhoods, it really all comes down to questions about balls.

Exercise
Given \( A \subseteq \mathbb{R}^p \) \& \( x \in \mathbb{R}^p \), prove TFAE.

a. \( x \) is a cluster point of \( A \).

b. \( \forall n \in \mathbb{N} \exists y_n \in A \text{ s.t. } 0 < \|x - y_n\| < \frac{1}{n}. \) (So, points of \( A \) really do "cluster" about \( x \).)
Example: \( A = \{ x \} \) has no cluster points, but it does have a boundary point (\( x \) itself).

**Exercise**

Consider \( A = \{ \frac{1}{n} : n \in \mathbb{N} \} \)

\[
\frac{1}{\frac{1}{2}} \frac{1}{\frac{1}{3}} \quad \frac{1}{\frac{1}{4}} \quad \frac{1}{\frac{1}{5}} \quad \frac{1}{\frac{1}{6}}
\]

a. Prove that \( x = 0 \) is the only cluster point of \( A \).

b. Prove that \( \bar{A} = A \cup \{ 0 \} \).

d. Show that no boundary points are cluster points.

c. Show that all interior points are cluster points.

d. Give examples showing that if \( x \in A \) and \( x \in \partial A \), then \( x \) may or may not be a cluster point of \( A \).
We can characterize closed sets by their cluster points, just as we did for boundary points.

**Exercise**

Let $F \subseteq \mathbb{R}^n$ be given. Then

$F$ is closed $\iff$ $F$ contains all its cluster points.

**Sketch for $\Rightarrow$**

Suppose $F$ is closed & $x$ is a cluster point.

If $x \notin F$, show that $x$ must be an exterior point, and that this implies it is not a cluster point.
Definition
A set $A \subseteq \mathbb{R}^p$ is bounded if it is contained in some open ball $B_r(x)$.

Exercise
Show $A$ is bounded if and only if $\exists R > 0$ s.t. $\|x\| < R \ \forall x \in A$, or, in other words, $A \subseteq B_R(0)$. 
Bolzano-Weierstrass Theorem

Every bounded infinite subset of \( \mathbb{R}^n \) has a cluster point.

Proof:

\( p=1 \)

Let \( B \subseteq \mathbb{R} \) be an infinite bounded set.

Then \( B \subseteq I_1 = [-c, c] \) for some \( c \). Let \( y_1 \) be one point in \( I_1 \).

\[
\begin{array}{c}
I_1 \\
B_L & \rightarrow & B_R \\
-\infty & \rightarrow & \infty
\end{array}
\]

one of these becomes \( I_2 \)

Let \( B_L = \text{left half of } I_1 \), \( B_R = \text{right half of } I_1 \).

End \( \)

Either \( B_L \) contains \( \infty \) many points of \( B \) or \( B_R \) (or both).

Let \( I_2 = \text{one with } \infty \text{ many points of } B \).

\( y_2 = \text{one point in } I_2, y_1 \neq y_2 \).
Repeat.

One half of \( I_2 \) contains \( \infty \) many points of \( B \).

\[ I_3 = \text{one half of } I_2 \text{ that contains } \infty \text{ many points of } B \]

\[ y_3 = \text{one point of } B \text{ in } I_3, \quad \neq y_1, y_2. \]

Etc.

\[ I_1 \supset I_2 \supset I_3 \supset \ldots \text{ nested cells.} \]

If \( y \in \bigcap_{n=1}^{\infty} I_n \) Nested cells property.

Note \( y \in I_1, \quad y \in I_i \quad \Rightarrow \quad ||y - y_i|| < 2r \)

\[ y \in I_2, \quad y_2 \in I_2 \quad \Rightarrow \quad ||y - y_2|| < r \]

\[ \vdots \]

\[ y \in I_n, \quad y_n \in I_n \quad \Rightarrow \quad ||y - y_n|| < \frac{r}{2^{n-2}}. \]

So \( \lim_{n \to \infty} ||y - y_n|| = 0 \).

And \( y \) can only equal at most one \( y_n \).

So \( y \) is a cluster point of \( B \). \( \Box \)
One subsquare becomes $I_2$, etc.