II. Heine-Borel Theorem

We start with the definition of a compact set.

**Definition**

A set $K \subseteq \mathbb{R}^p$ is **compact** if:

Whenever $K$ is contained in the union of any open sets $G_\alpha$ ($\alpha$ running through some index set $I$), then $K$ must be contained in the union of finitely many of these $G_\alpha$.

In other words:

$$K \subseteq \bigcup_{\alpha \in I} G_\alpha, \text{ } G_\alpha \text{ open} \Rightarrow \exists \alpha_1, ..., \alpha_n \text{ s.t. } K \subseteq \bigcup_{k=1}^n G_{\alpha_k}$$

**Note**

This is a conditional statement. It does not say that $K$ is compact if it can be covered by finitely many open sets. Instead, it says something much more involved: If you can cover $K$ by open sets, then no matter what this covering is, you can select...
finitely many of these open sets that still cover $K$.

Every cover of $K$ by open sets has

a finite subcover.

Again: not just some cover but every possible
open cover must have a finite subcover.

Example
Suppose that $K = \{x_1, \ldots, x_n\}$ is a finite set.

Suppose that $G_x, x \in I$ are any open sets such that

$K \subseteq \bigcup_{x \in I} G_x$. Then $x_1 \in K \subseteq \bigcup_{x \in I} G_x$, so there

is some $x_1$ s.t. $x_1 \in G_{x_1}$. Likewise $x_2$ belongs to

some $G_{x_2}$, etc. Hence $K \subseteq G_{x_1} \cup \cdots \cup G_{x_n}$,

i.e., $K$ is covered by finitely many of the $G_x$,

no matter what sets $G_x$ we chose to cover

$K$ with.
Showing that a set is compact can be hard: you have to prove that every possible open cover has a finite subcover.

What do we have to do to prove that a set $A$ is not compact? We have to prove that it's not true that every open cover has a finite subcover. To do that we only have to show that there exists one covering of $A$ by open sets $G_\alpha$ s.t. if we only use finitely many of the $G_\alpha$ then $A$ won't be covered.

*Example:* $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$ is not compact.

To prove this, we have to find a cover of $A$ by open sets that no longer covers $A$ if only finitely many of those open sets are used. There
may be other cases that have finite subcovers, but that doesn't matter. For example, we can cover set $A$ with one open set:

$$A \subseteq (0, 2).$$

This does not show $A$ is compact & it does not show $A$ is not compact.

Instead, set $G_i = \left( \frac{1}{i}, 2 \right)$ and

$$G_n = \left( \frac{1}{n+1}, \frac{1}{n-1} \right) \text{ for } n \geq 2.$$ Then each $G_n$ is open & $\frac{1}{n} \in G_n$ for each $n \in \mathbb{N}$, so

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \bigcup_{n=1}^{\infty} G_n.$$ However, each point $\frac{1}{n}$ belongs to one and only one set $G_k$, namely $G_n$. Hence if we only use finitely many $G_n$, say $G_{n_1}U \ldots U G_{n_k}$, then we'll only cover
The points $n_1, \ldots, n_k$ and there will be infinitely many points in $A$ that are not covered. Thus, this particular open cover $\{G_n\}_{n \in \mathbb{N}}$ has no finite subcover. Therefore $A$ is compact.

**Notation**

If $E_x, x \in I$ are sets s.t.

$$A \subseteq \bigcup_{x \in I} E_x$$

then we call $\{E_x\}_{x \in I}$ a cover of $A$.

If each $E_x$ is open, then it is an open cover.

If $J \subseteq I$ & $\{E_x\}_{x \in J}$ still covers $A$,

then we call it a subcover. If $J$ is finite

then it is a finite subcover.
Example The interval \( A = (0, 1) \) is not compact.

Again we just have to find one cover of \( A \) by open sets that has no finite subcover. Consider

\[ G_n = \left( \frac{1}{n}, 1 \right) \]

Each \( G_n \subseteq (0, 1) \), so we certainly have

\[ \bigcup_{n=1}^{\infty} G_n \subseteq (0, 1) \].

On the other hand, if \( x \in (0, 1) \) then \( 0 < x < 1 \), so \( \exists n \in \mathbb{N} \) s.t. \( \frac{1}{n} < x < 1 \) (why?)

Hence \( x \in G_n \), so \( (0, 1) \subseteq \bigcup_{n=1}^{\infty} G_n \). Thus \( \{G_n\}_{n=1}^{\infty} \) is an open cover of \((0, 1)\). But if we only use finitely many \( G_n \), say \( G_{n_1}, \ldots, G_{n_k} \), then we cannot cover \((0, 1)\).

Why? By reordering, we can take \( n_1 < \cdots < n_k \).

Then no \( x \) between 0 & \( \frac{1}{n_k} \) will be covered by \( G_{n_1}, \ldots, G_{n_k} \) (why?). We can't cover all of \((0, 1)\) using finitely many \( G_n \), so \((0, 1)\) is not compact.
We will shortly see some better ways of dealing with compact sets in $\mathbb{R}^p$, but for now we use the definition of compactness to show the following.

**Exercise**
Suppose that $K$ is a compact subset of $\mathbb{R}^p$. Choose any $r > 0$. Show finitely many points $x_1, \ldots, x_n \in \mathbb{R}^p$ s.t.

$$K \subseteq \bigcup_{k=1}^n B_r(x_k).$$

**Hint:** Consider all of the balls!

Next we show that in $\mathbb{R}^p$, a set is compact if & only if it is closed & bounded.

**Beware:** Compact, open, & closed sets make sense in much more general topological spaces, and it is not true in an arbitrary topological space that compact = closed & bounded!!
Heine-Borel Theorem: Let $F \subseteq \mathbb{R}^p$ be given. Then:

$F \subseteq \mathbb{R}^p$ is compact $\iff$ $F$ is closed & bounded.

Proof:

$\Rightarrow$ Assume $F$ is compact.

Let $G_n = B_n(0)$. Then $F \subseteq \mathbb{R}^p = \bigcup_{n \in \mathbb{N}} G_n$.

Since $F$ is compact, it must be contained in a finite union of $\& G_n$. But $\& \text{F is contained in some ball } B_n(0)$, so is bounded.

Next, we show $F$ is closed by showing that $\overline{C(F)}$ is open.

Let $x \in \overline{C(F)}$. Let $G_n = \{ y \in \mathbb{R}^n : \| x - y \| > \frac{1}{n} \} = \overline{C(B_{\frac{1}{n}}(x))}^c \subseteq F$.

Note: $G_n$ is open, & $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{R} \setminus \{0\} = F$. 
Since \( F \) is compact, \( F \subseteq G_i \subseteq U \cdots \subseteq U G_N \) for some \( N \). But \( G_i \subseteq G_i \subseteq \cdots \subseteq G_N \), so \( F \subseteq G_N \). Hence

\[
\exists \quad \text{open ball}
\]

Thus \( \overline{B}_{\gamma N}(x) \subseteq E(F) \)

But \( \forall x \in E, \overline{B}_{\gamma N}(x) = E(F) \)

so \( E(F) \) is open.
Assume $F$ is closed & bounded.

Let $\{G_x\}$ be any open cover of $F$ (countable or uncountable). Suppose there is no finite subcover of $\{G_x\}$.

Since $F$ is bounded, $F \subseteq \text{cubes } I_k$ for some $r$.

One of $F \cap I_k$, $F \cap I_k'$, $F \cap I_k''$, $F \cap I_k'''$ cannot be covered by finitely many $G_x$. Call this one $I_k$. Repeat.
$I_1 \subseteq I_2 \ldots$ Nested cells. $I \subseteq \mathbb{N}$

Note $y$ is a cluster point of $F$. But $F$ is closed, so $y \in F$.

Hence $y \in G_\varepsilon$ for some $\varepsilon$. But then $B_\varepsilon(y) \subseteq G_\varepsilon$ for some $\varepsilon$.

But $I_k \subseteq B_\varepsilon(y)$ for all $k \geq$ some $k_0$.

Hence $F \cap I_k$ is covered by the single set $G_\varepsilon$ — contradiction.

\[ \text{dist}(x, F) = \|x - y\| > 0 \]

\[ \text{dist}(x, u) = 0 \]
Cantor Intersection Theorem

If $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ are (closed) & nonempty, then $\cap F_n \neq \emptyset$.

Proof:
Let $G_n = c(F_n)$. Suppose $\cap F_n = \emptyset$. Then $\cap G_n = \emptyset$. Hence $F_1 \supseteq \cup G_n$. But $F_1$ is closed & bounded, hence compact.

So $F_1 \supseteq G_1 \cap \ldots \cap G_n = G_n = c(F_n)$.

$F_i \cap F_n = \emptyset$ because $F_i \cap F_n \subseteq c(F_n) \cap F_n = \emptyset$

Contradiction: $F_n \subseteq F_i$ !

Example: If $F_n$ aren't bounded, could have $\cap F_n = \emptyset$.
Let $F_n = c(B_n(0)) = \{ y \in \mathbb{R}^p : \| y \| \geq n \}$.

Nested: $F_1 \supseteq F_2 \supseteq \ldots$ but $\cap F_n = \emptyset$
Corollary

Let $F \subseteq \mathbb{R}^p$ be closed, $F \neq \emptyset$. Let $x \notin F$.

Then there exists a point $y \in \overline{B}(x, \varepsilon)$ that is nearest to $x$, i.e.,

$$
\forall z \in F \Rightarrow \|z - x\| \geq \|z - y\|.
$$

Proof

Let

$$
d = \inf \{ \|z - x\| : z \in F \},
$$

We claim that $d > 0$. Suppose $d = 0$. Then by def. of inf, $\exists n \in \mathbb{N}$,

$$
\exists z_n \in F \text{ s.t. } 0 < \|z_n - x\| < \frac{1}{n}.
$$

Hence $x$ is a cluster point of $F$.

But $F$ is closed, so this implies $x \in F$, a contradiction.

Let $F_n = \overline{B}(x, \frac{d}{n}) \cap F$. $F \supseteq F_2 \supseteq \ldots$.

By Cantor Intersection, $\exists y \in \bigcap_{n=1}^{\infty} F_n$. $\|y - x\| \leq \frac{d}{n}$ $\forall n$.

Note $y \in F$. So $\|y - x\| \leq d$.

But $d = \inf \{ \|x - z\| : z \in F \}$ so $\|y - x\| \leq \|x - z\| \forall z \in F$.

Note: There can be more than one closest point.