11.

<u>Heine-Borel Theorem</u> Let $F \subseteq \mathbb{R}^p$ be given. Then: 2

$F \subseteq \mathbb{R}^p$ is compact $\iff$ F is closed & bounded.

<u>Proof:</u>

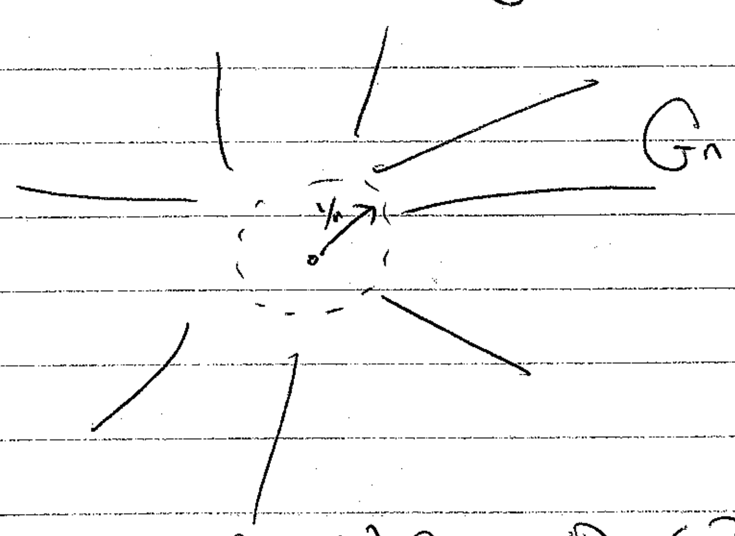
$\Rightarrow$ Assume F is compact.

Let $G_n = B_n(0)$. Then $F \subseteq \mathbb{R}^p = \bigcup_{n \in \mathbb{N}} G_n$.

Since F is compact, it must be contained in a finite union of ~~&~~ G_n. But ~~&~~ F is contained in some ball $B_n(0)$, so is bounded.

Next, we show F is closed by showing that $\mathcal{C}(F)$ is open.

Let $x \in \mathcal{C}(F)$. Let $G_n = \{y \in \mathbb{R}^n : \|x-y\| > \frac{1}{n}\}$.
$= \mathcal{C}(\overline{B_{1/n}(x)})$

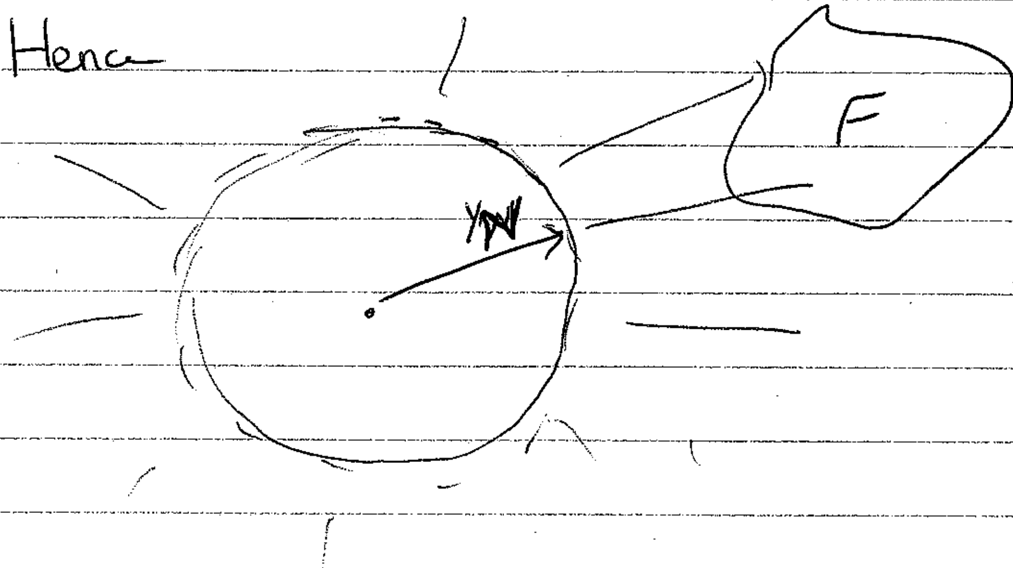


Note G_n is open, & $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{R} \setminus \{0\} \supseteq F$.

Since F is compact, $F \subseteq G_1 \cup \dots \cup G_N$

for some N. But $G_1 \subseteq G \subseteq \dots \subseteq G_N$, so

$F \subseteq G_N$. Hence



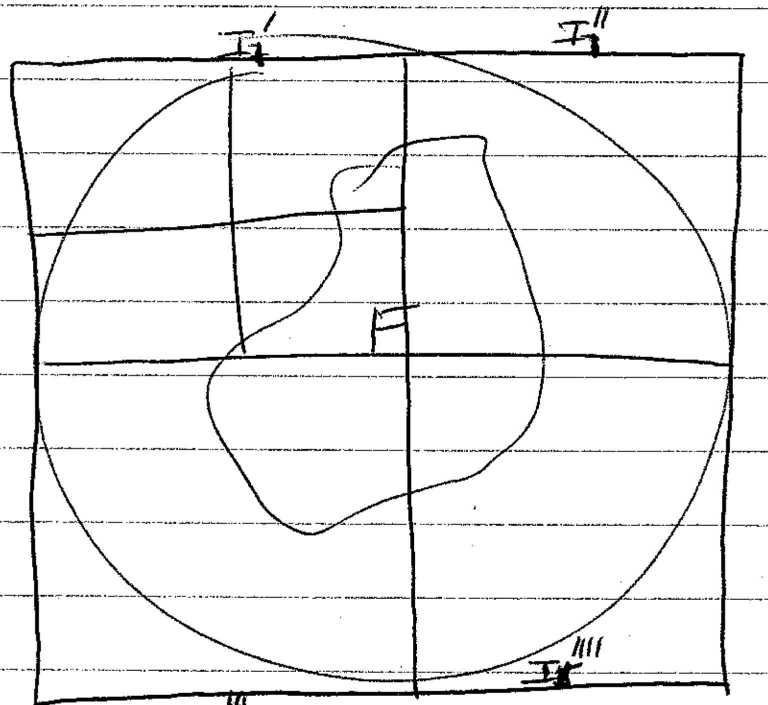
Thus $\bar{B}_{1/N}(x) \subseteq \mathcal{C}(F)$. But then $x \in B_{1/N}(x) \subseteq \mathcal{C}(F)$
open ball

so $\mathcal{C}(F)$ is open.

$\Leftarrow$ Assume F is closed & bounded.

Let $\{G_\alpha\}$ be <u>any</u> open cover of F (countable or uncountable). Suppose there is <u>no</u> finite subcover of $\{G_\alpha\}$.

Since F is bounded, $F \subseteq$ Cube I_1 for some r.



$\overline{B_r(0)} \subseteq I_1$

One of $F \cap I_1'$, $F \cap I_1''$, $F \cap I_1'''$, $F \cap I_1''''$ cannot be covered by finitely many G_α. Call this one I_2.

Repeat

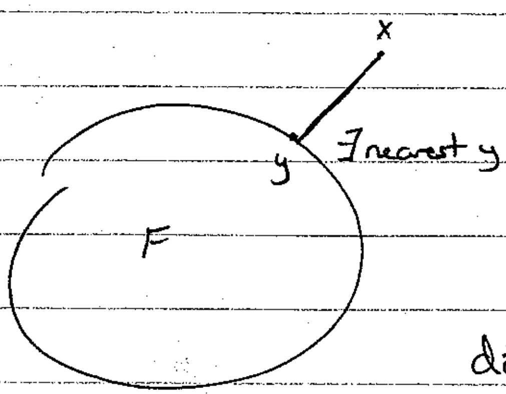
$I_1 \supseteq I_2 \supseteq \cdots$ Nested cells. $\exists\, y \in \bigcap I_n$

Note y is a cluster point of F. But F is closed, so $y \in F$.

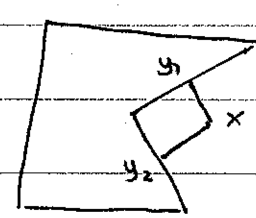
Hence $y \in G_\alpha$ for some α. But then $B_\varepsilon(y) \subseteq G_\alpha$ for some ε.

But $I_k \subseteq B_\varepsilon(y)$ for all $k \geq$ some k_0.

Hence $F \cap I_k$ is covered by the single set G_α — contradiction. $\blacksquare$



need not be unique



$\operatorname{dist}(x, F) = \|x - y\| > 0$

$\operatorname{dist}(x, u) = 0$

Cantor Intersection Theorem

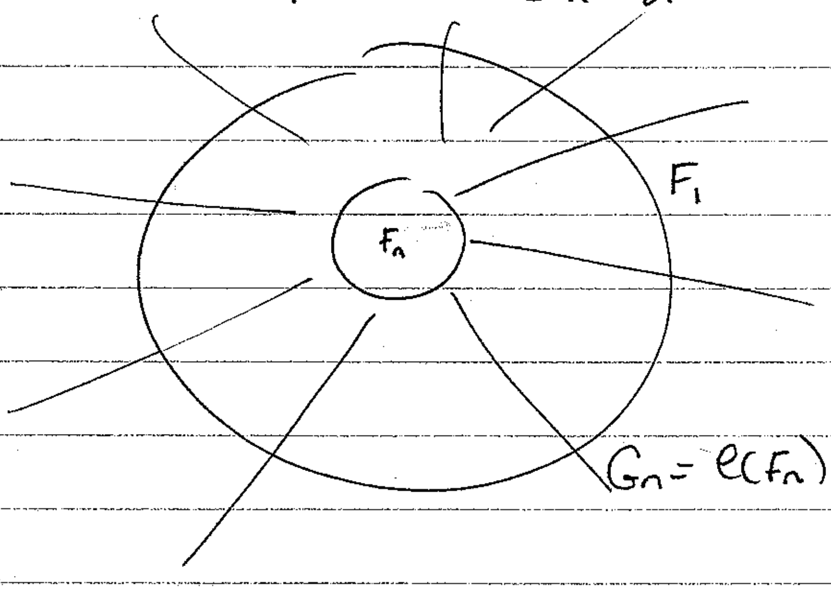
If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ are $\left(\begin{smallmatrix}\text{compact} \\ \text{closed} \\ \text{bounded}\end{smallmatrix}\right)$ & nonempty, then $\bigcap F_n \neq \emptyset$.

Proof.

Let $G_n = \mathcal{C}(F_n)$. Suppose $\bigcap F_n = \emptyset$. Then $\bigcup G_n = \mathbb{R}^p$.

Hence $F_1 \subseteq \bigcup_n G_n$. But F_1 is closed & bounded, hence compact.

So $F_1 \subseteq G_1 \cup \cdots \cup G_n = G_n = \mathcal{C}(F_n)$.



$F_1 \cap F_n = \emptyset$

because

$F_1 \cap F_n \subset \mathcal{C}(F_n) \cap F_n = \emptyset$

Contradiction: $F_n \subseteq F_1$! $\boxtimes$

Example: If F_n aren't bounded, could have $\bigcap F_n = \emptyset$.

Let

$$F_n = \mathcal{C}(B_n(0)) = \{y \in \mathbb{R}^p : \|y\| \geq n\}.$$

Nested: $F_1 \supseteq F_2 \supseteq \cdots$ but $\bigcap F_n = \emptyset$

Corollary

Let $F \subseteq \mathbb{R}^p$ be closed, $F \neq \emptyset$. Let $x \notin F$.

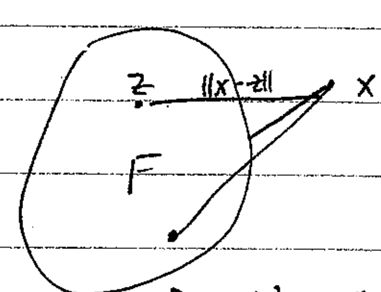
Then $\exists$ a point $y \in F$ that is nearest to x, i.e.

$$z \in F \implies \|z-x\| \geq \|z-y\|.$$

Proof:

Let

$$d = \inf\{\|z-x\| : z \in F\}. \quad = \operatorname{dist}(x,F)$$



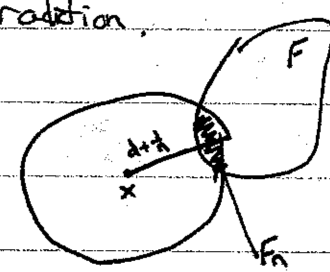
We claim that $d > 0$. Suppose $d = 0$. Then by def. of inf, $\forall n \in \mathbb{N}$,

$\exists z_n \in F$ s.t. $0 < \|z_n - x\| < \frac{1}{n}$. Hence x is a cluster point of F.

$\uparrow$ because $x \notin F$

But F is closed, so this implies $x \in F$, a contradiction.

Let $F_n = \overline{B}_{d+\frac{1}{n}}(x) \cap F \quad F_1 \supseteq F_2 \supseteq \dots$



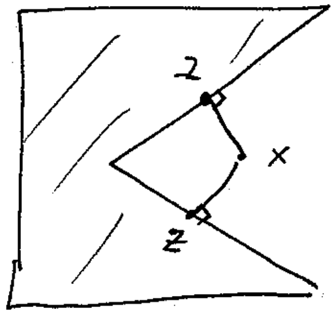
By Cantor Intersection, $\exists y \in \bigcap_{n=1}^{\infty} F_n$. $\quad \|y-x\| \leq d + \frac{1}{n} \;\forall n$

Note $y \in F$. So $\|y-x\| \leq d$.

But $d = \inf\{\|x-z\| : z \in F\}$ so $\|y-x\| \leq \|x-z\| \quad \forall z \in F$ $\square$

Note: There can be more than one closest point

Example:

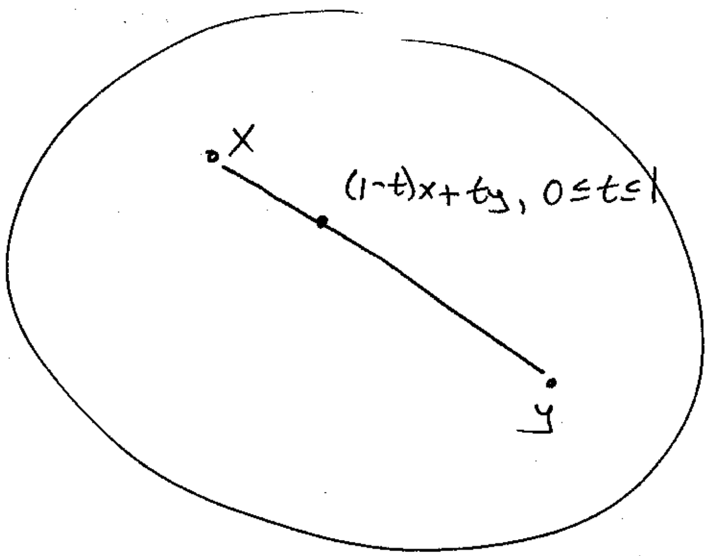


K is compact, but there are two points closest to x. Problem: K is not convex.

Exercise: K closed & convex $\Rightarrow$ unique closest point.

Hint: If there were two, the midpoint would be even closer.

Convex:



K convex if:

$x, y \in K \Rightarrow (1-t)x + ty \in K$ for $0 \leq t \leq 1$.