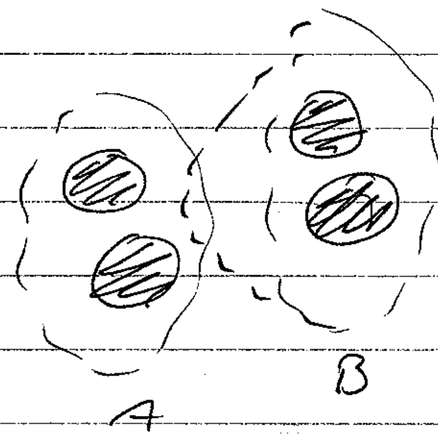
12 Connected Sets

Definition

$D \subseteq \mathbb{R}^p$ is disconnected if



$\exists$ nonempty open A, B st.

$$A \cap D \neq \emptyset, \quad B \cap D \neq \emptyset$$

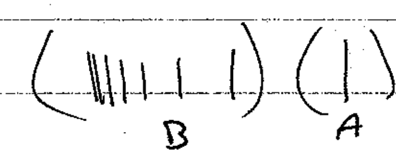
$$(A \cap D) \cap (B \cap D) = \emptyset$$

$$(A \cap D) \cup (B \cap D) = D$$

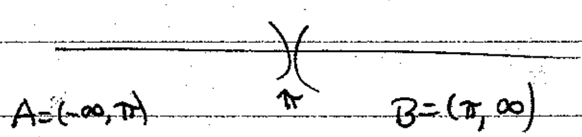
Then (A, B) is a disconnection of D.

$C \subseteq \mathbb{R}^p$ is connected if it is not disconnected.

Ex: $\{\frac{1}{n} : n \in \mathbb{N}\}$ is disconnected.



Ex: $\mathbb{Q}$ is disconnected



Ex: $(0,1)$ is connected

$[0,1)$

$(0,1]$

$[0,1]$

Exercise: Connected, x clusterpt $\Rightarrow C \cup \{x\}$ connected

Suppose (A,B) was a disconnection of $C \cup \{x\}$.

Show it would then be a disconnection of C.

Theorem: Any interval in $\mathbb{R}$ is connected.

Proof: Suppose $I=(0,1)$ is disconnected, let (A,B) be a disconnection.

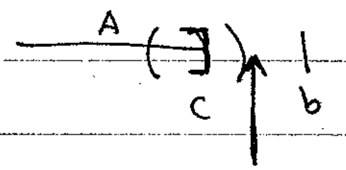
$I\cap A$, $I\cap B$ are not single points (they are open).

Let $a\in A$, $b\in B$. Either $a<b$ or $b<a$.

Assume $0<a<b<1$. Let $c = \sup\{x\in A : x<b\}$.

Note $a \leq c \leq b$ so $c\in I = A\cup B$.

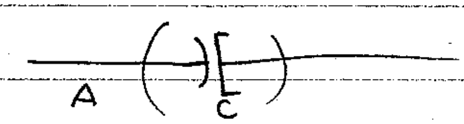
If $c\in A$ then $c\neq b$.



A is open! $(c-\varepsilon, c+\varepsilon)\subseteq A$

Contradiction.

If $c\in B$ then:



$(c-\varepsilon, c+\varepsilon)\subseteq B$

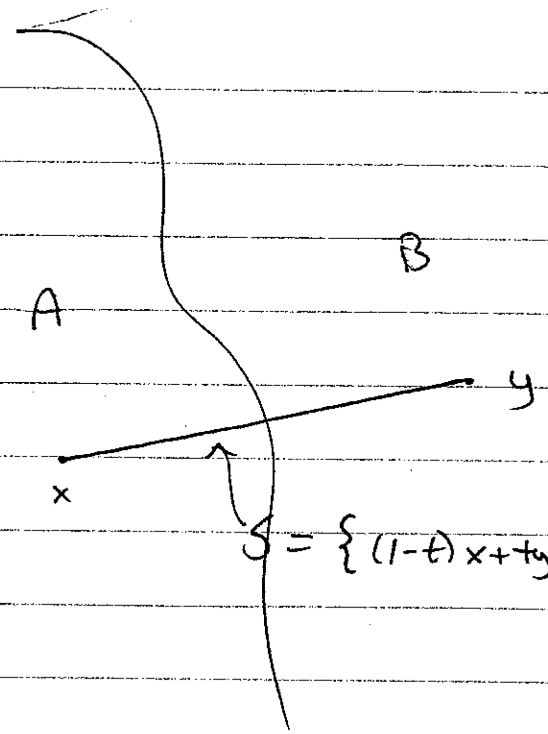
B is open! Contradiction. $\blacksquare$

Ex. $\mathbb{R}^p$ is connected.

Proof:

Suppose not

Choose $x \in A$, $y \in B$.

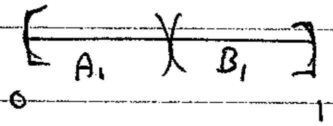


$S = \{(1-t)x + ty : 0 \le t \le 1\}$

Let

$A_1 = \{t : (1-t)x + ty \in A\}$

$B_1 = \{t : (1-t)x + ty \in B\}$



Exercise: (A_1, B_1) is a disconnection of $[0,1]$.

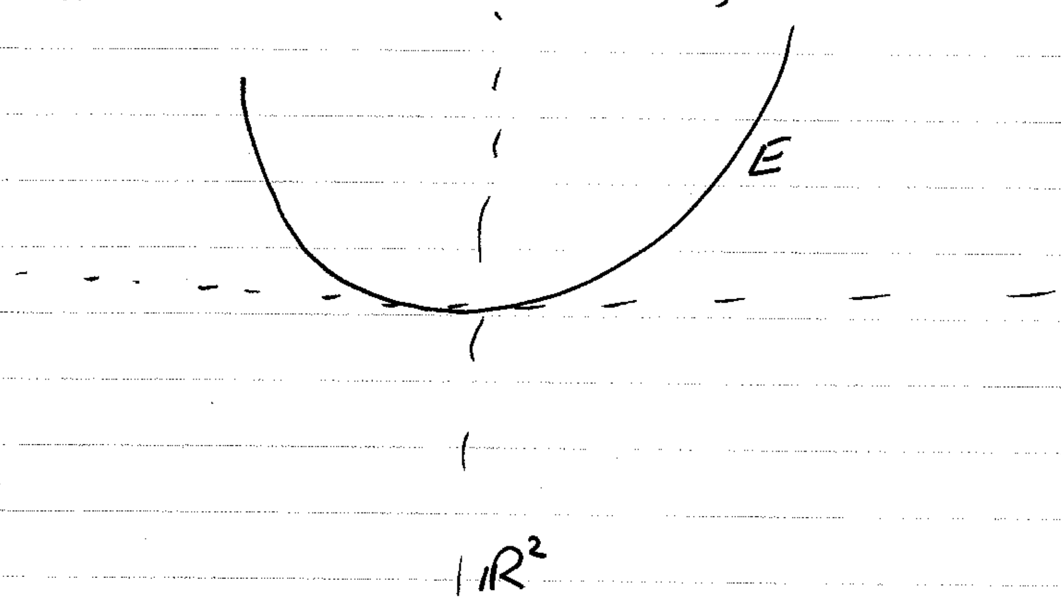
$\blacksquare$

Corollary Only $\emptyset$, $\mathbb{R}^p$ can be both open & closed.

Proof:

Suppose A is both open & closed. Then $e(A)$ is both open & closed. If $A \neq \emptyset$, $e(A) \neq \emptyset$ then $(A, e(A))$ forms a disconnection of $\mathbb{R}^p$, which is impossible. $\blacksquare$

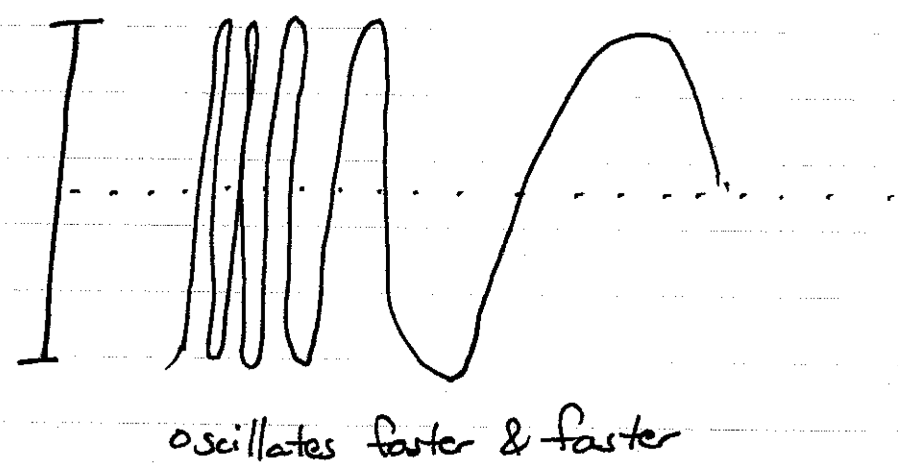
Example: $E = \{(x, x^2) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$



E is connected by not polygonally path-connected.

Perhaps it be better to use any "continuous" curve to define path-connected. But consider:

Example: $E = \{(0,y) : -1 \le y \le 1\} \cup \{(x, \sin\frac{1}{x}) : x > 0\}$

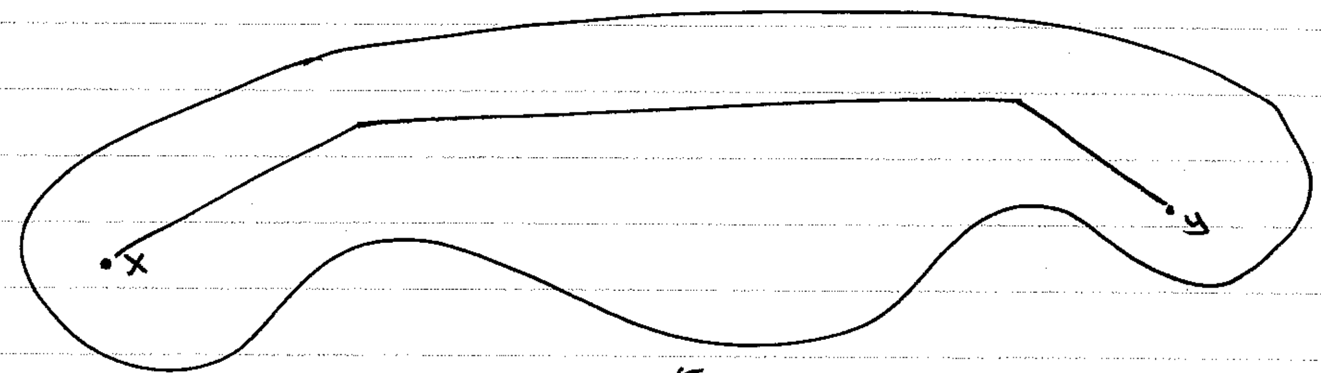


Exercise: E is connected!

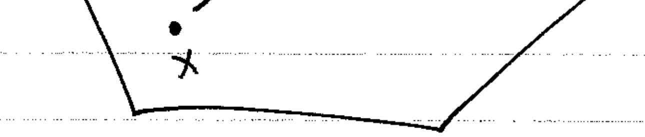
But clearly E is not path-connected, even using any continuous curves.

Definition

A set $E \subseteq \mathbb{R}^n$ is polygonally path-connected if every $x, y \in E$ can be joined by a polygonal curve lying entirely within E.



Example: All convex sets are polygonally path-connected.



convex means any two $x, y \in E$ can be joined by a line segment

Q. Are all connected sets polygonally path-connected?

Are all polygonally path-connected sets connected?

Theorem

If $G \subseteq \mathbb{R}^p$ is open then

G is connected $\iff$ G is polygonally path-connected.

Proof:

$\Rightarrow$ Assume G is connected & choose any $x \in G$.

Set

$$G_1 = \{y \in G : y \text{ can be joined to } x \text{ by a polygonal path contained in } G\}$$

$$G_2 = \{z \in G : z \text{ cannot be joined to } x \text{ by a polygonal path contained in } G\}.$$

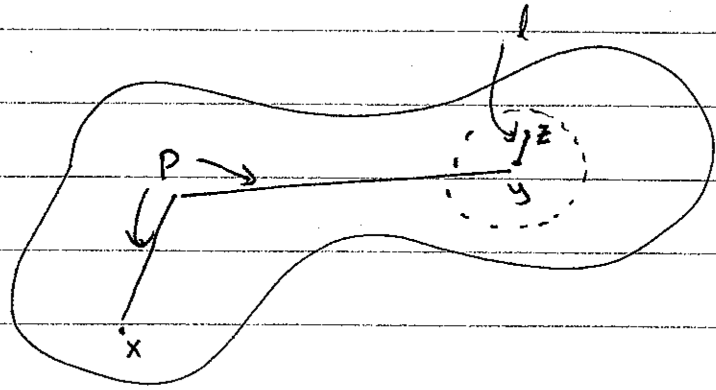
We must show that $G_2 = \emptyset$.

Note that $G_1 \cup G_2 = G$ & $G_1 \cap G_2 = \emptyset$.

Claim: G_1 is open.

To see this, choose $y \in G_1$. Since G is open, $\exists\, r > 0$ s.t. $B_r(y) \subseteq G$. We will show $B_r(y) \subseteq G_1$. Choose $z \in B_r(y)$. We must show $z \in G_1$.

Let P be the polygonal curve joining x to y.



Note y is joined to z by a line segment ℓ lying in G.
Hence $P \cup \ell$ is a polygonal curve joining x to z.
Hence $z \in G_1$. So G_1 is open.

Claim: G_2 is open. Exercise.

Thus (G_1, G_2) is a disconnection of G if $G_1, G_2 \neq \emptyset$.
Hence one of G_1, G_2 must be empty. Since $x \in G_1$, it can't be empty. Therefore $G_2 = \emptyset$.

$\Leftarrow$ ~~We'll prove the contrapositive: If G is not connected then $\exists\, x, y \in G$ that cannot be joined by a polygonal curve lying in G.~~

~~Assume G is not connected. Then $\exists$ disconnection (A, B).~~

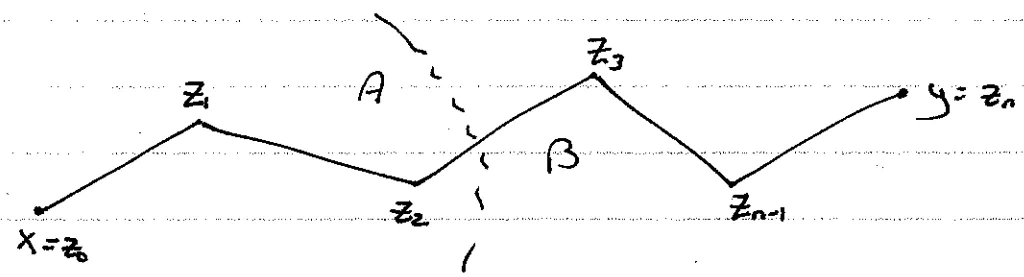
~~Let $x \in A \cap G$ & $y \in B \cap G$.~~

$\Leftarrow$ Suppose every $x, y \in G$ can be joined by a polygonal curve in G.

Suppose G was disconnected & let (A, B) be a disconnection.

Let $x \in A \cap G$, $y \in B \cap G$. Let $z_0 = x$, $z_1, \ldots, z_{n-1}$, $z_n = y$

be the endpoints of the line segments joining x to y.



Let z_k be the last endpoint lying in A: note $k \leq n-1$.

So $z_{k+1} \in B$. Define

$$A_1 = \{ t \in [0,1] : (1-t) z_k + t z_{k+1} \in A \cap G \}$$
$$B_1 = \{ t \in [0,1] : \qquad\qquad \in B \cap G \}$$

Then (A_1, B_1) is a disconnection of $[0,1]$, which is a contradiction. $\blacksquare$

13. Complex Numbers

Read on your own. $\exists$ multiplication of complex nos.

$\nexists$ ordering of $\mathbb{C}$