14. Sequences

in a set

A sequence (with values in a set) S is a function from $\mathbb{N}$ to S.

$$X : \mathbb{N} \longrightarrow S \qquad\qquad X = (x_1, x_2, \ldots) = (x_n)$$
$$n \longmapsto X(n) = x_n$$

We will deal with sequences whose values are in $\mathbb{R}^p$.
MOST RESULTS HOLD FOR ANY SPACE THAT HAS A NORM.

Sum of sequences $X = (x_n)$ & $Y = (y_n)$ is $X + Y = (x_n + y_n)$

$\boxed{\text{Componentwise addition}}$

Scalar Mult. $cX = (cx_n)$

$V = \{$all sequences $X = (x_n) \;:\; x_n \in \mathbb{R}^p\}$ is a vector space.

V is more than a vector space

Products: $XY = (x_n y_n)$ $\qquad$ V is an algebra w.r.t. componentwise multiplication.

Quotients: $X/Y = (x_n / y_n)$ But not always defined!

"Inner Product" (Bartle notation) $X \cdot Y = (x_n \cdot y_n)$

Beware: INCONSISTENT NOTATION!
This is not an inner product on V! $X \cdot Y$ is a sequence, not a real number!

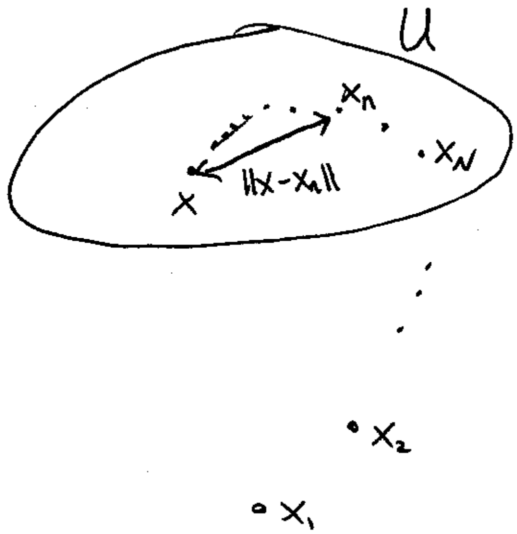
Definition: Limit of an $\mathbb{R}^p$-valued sequence (valid for any normed space)

Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$.

Then $x \in \mathbb{R}^p$ is a limit of X if

$\forall$ neighborhood U of x, $\exists N > 0$ s.t.

$$n \geq N \implies x_n \in U.$$



From some point onwards, you have to say in U.

As usual when dealing with neighborhoods, everything comes down in the end to open balls.

Exercise

Given $X = (x_n)$ & $x \in \mathbb{R}^p$, show TFAE.

a. x is a limit of $X = (x_n)$.

b. $\forall \varepsilon > 0 \ \exists N > 0$ s.t. $n \geq N \Rightarrow \|x - x_n\| < \varepsilon$.

c. $\forall k \in \mathbb{N} \ \exists N > 0$ s.t. $n \geq N \Rightarrow \|x - x_n\| < \frac{1}{k}$.

Notation

We write any of

$$x_n \longrightarrow x$$

or $$x = \lim_{n\to\infty} x_n = \lim_n x_n = \lim x_n = \lim X$$

to mean that x is a limit of $X = (x_n)$.

Exercise

a. $X = \left(\frac{1}{n}\right)$ is convergent (has a limit)

b. $X = \left((-1)^n\right)$ is not convergent (has no limit).

Next we give another "Highlander Theorem."

Lemma (Valid for any normed space).

A sequence $X = (x_n)$ in $\mathbb{R}^p$ can have at most one limit.

Proof:

Suppose $x_n \to x$ & $x_n \to y$. Choose any $\varepsilon > 0$.

Then:

$$\exists N_1 \text{ s.t. } n \geq N_1 \Rightarrow \|x - x_n\| < \varepsilon.$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \Rightarrow \|y - x_n\| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then we have both $n \geq N_1$ & $n \geq N_2$, so

$$\begin{aligned} \|x-y\| &= \|x - x_n + x_n - y\| \\ &\leq \|x - x_n\| + \|x_n - y\| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

Thus $0 \leq \|x-y\| < 2\varepsilon$ for every $\varepsilon > 0$, so $\|x-y\| = 0$ & therefore $x = y$. ∎

Definition

$X = (x_n)$ is <u>bounded</u> if $\exists M > 0$ s.t. $\|x_n\| \leq M \ \forall n \in \mathbb{N}$.

Exercise

$X = (x_n)$ is bounded if & only if $\sup_n \|x_n\| < \infty$.

Lemma (Valid for any normed space).

All convergent sequences are bounded.

Proof:

Suppose $x_n \longrightarrow x$. Choose $\varepsilon = 1$. Then $\exists N$ s.t.

$$n \geq N \implies \|x - x_n\| < 1.$$

Hence $x_n \in B_1(x)$ for $n \geq N$, & for these n,

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$$
$$< 1 + \|x\|.$$

This is true for $n \geq N$ only. Therefore if n is <u>any</u> positive integer then

$$\|x_n\| \leq \max\{\|x_1\|, \dots, \|x_N\|, \ \|x\| + 1\} \quad \text{(why?)}$$

So X is bounded. $\blacksquare$

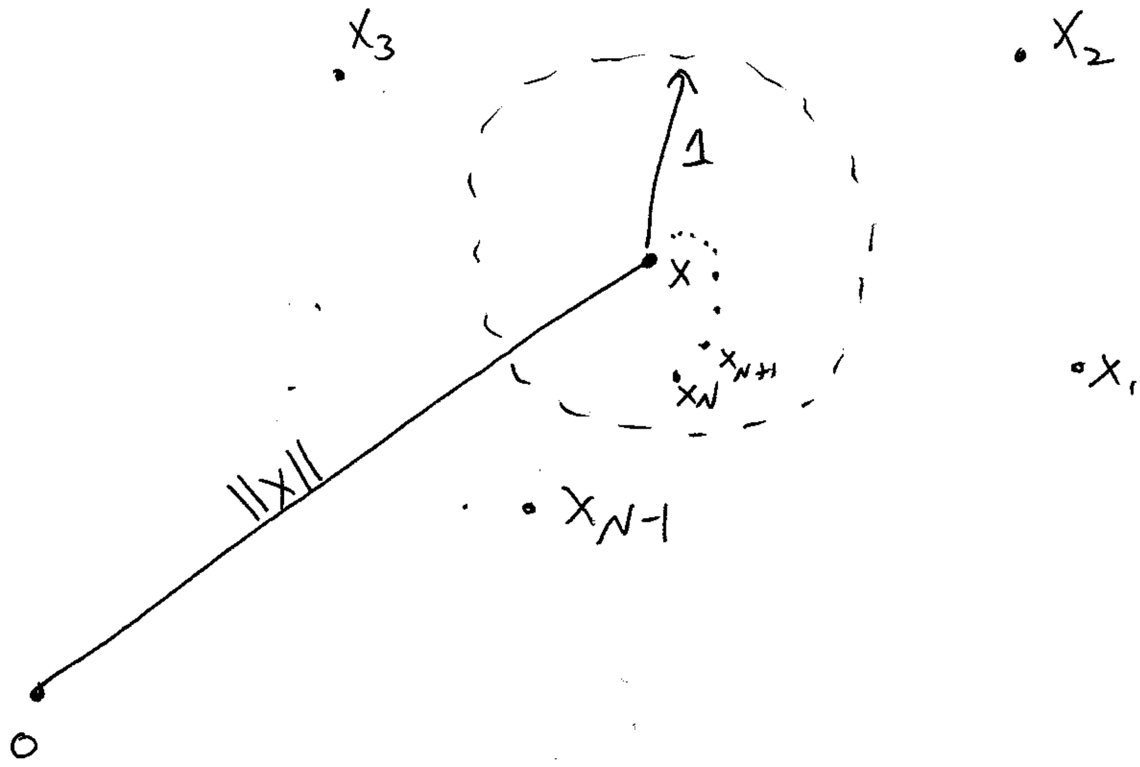


Illustration for proof of lemma.

<u>Exercise</u>: Not all bounded sequences converge!

<u>Definition</u>

Given a sequence $X = (x_n)$ of points in $\mathbb{R}^p$, write the components of x_n as

$$x_n = (x_{n,1}, x_{n,2}, \dots, x_{n,p})$$

Given $y = (y_1, \dots, y_p) \in \mathbb{R}^p$, we say that

x_n <u>converges componentwise</u> to y if

$$\lim_{n\to\infty} x_{n,k} = y_k \quad \text{for} \quad k = 1, \dots, p.$$

<u>Theorem</u> ($\Rightarrow$ holds more generally, $\Leftarrow$ is because $\mathbb{R}^p$ is finite-dimensional)

Given a sequence $X = (x_n)$ in $\mathbb{R}^p$ & given $y \in \mathbb{R}^p$,

$$\underset{\text{(convergence of vectors)}}{x_n \longrightarrow y} \quad \Longleftrightarrow \quad x_n \text{ converges componentwise to } y$$

<u>Idea of proof:</u>

$\Longrightarrow$ For each k,

$$|x_{n,k} - y_k| \le \|x_n - y\|$$

If $x_n \to y$ then RHS $\longrightarrow 0$.

$$\Longleftarrow \quad \|x_n - y\| = \sqrt{(x_{n,1}-y_1)^2 + \cdots (x_{n,p}-y_p)^2}$$

Componentwise convergence implies that the RHS $\to 0$

(important: there are only finite many components!)

Exercise: Make this proof sketch precise.

Some Facts about sequences

$\lim_{n\to\infty} x_n = x$ [limit of vectors x_n in $\mathbb{R}^n$]

$\iff \forall \varepsilon>0\ \exists N>0$ s.t. $n \geq N \Rightarrow \|x-x_n\| < \varepsilon$

$\iff \forall \varepsilon>0\ \exists N>0$ st. $n \geq N \Rightarrow \big|\|x-x_n\| - 0\big| < \varepsilon$

$\iff \lim_{n\to\infty} \|x-x_n\| = 0$ [limit of numbers $\|x-x_n\|$ is zero]

Remark: $n>N$ or $n\geq N$: doesn't matter
$\|x-x_n\|<\varepsilon$ or $\|x-x_n\|\leq\varepsilon$ doesn't matter } exercise

Lemma: Convergence of sums
$x_n \to x$ & $y_n \to y \Rightarrow x_n+y_n \to x+y$

Same with different notation:

$\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty}(x_n+y_n) = x+y$

Exercise: Converse is false! Give a counterexample!

Proof of Lemma Suppose $x_n \to x$ & $y_n \to y$.
Choose $\varepsilon>0$. Then $\varepsilon/2>0$, so

$\exists N_1$ s.t. $n>N_1 \Rightarrow \|x-x_n\| < \varepsilon/2$

$\exists N_2$ s.t. $n>N_2 \Rightarrow \|y-y_n\| < \varepsilon/2$

Let $N = \max\{N_1, N_2\}$. If $n > N$ then we have both $n > N_1$ & $n > N_2$, so

$$\begin{aligned} \|(x+y) - (x_n + y_n)\| &\leq \|x - x_n\| + \|y - y_n\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $x_n + y_n \longrightarrow x + y$. ∎

<u>Lemma</u> Convergence of products.

$$x_n \to x, \quad y_n \to y \quad \Longrightarrow \quad x_n y_n \longrightarrow xy$$

Note: These are componentwise products:

$$x_n y_n = (x_{n,1} y_{n,1}, \ldots, x_{n,p} y_{n,p}).$$

<u>Proof sketch for $p=1$</u>

When $p=1$, x_n, y_n, x, y are all <u>numbers</u>, so $x_n \to x$, $y_n \to y$ means that $|x - x_n| \to 0$, $|y - y_n| \to 0$ (absolute value). Then

$$|xy - x_n y_n| = |xy - x_n y + x_n y - x_n y_n|$$

$$\leq |xy - x_n y| + |x_n y - x_n y_n|$$

$$\leq \underbrace{|y|}_{\text{constant}} \underbrace{|x - x_n|}_{\substack{\text{small} \\ \text{for } n \text{ large}}} + \underbrace{|x_n|}_{?} \underbrace{|y - y_n|}_{\substack{\text{small} \\ \text{for } n \text{ large}}} \qquad (*)$$

Choose ε. $\exists N_1$ s.t.

$$n > N_1 \implies |x - x_n| < \frac{\varepsilon}{2|y|} \qquad \text{(What if } y = 0\text{??)}$$

So the first term in $(*)$ will be $< \frac{\varepsilon}{2}$ for n large.

For the second term, recall that convergence sequences are bounded. So $\exists M$ s.t.

$$|x_n| \leq M \quad \forall n.$$

Then $\exists N_2$ s.t.

$$n > N_2 \implies |y - y_n| < \frac{\varepsilon}{2M}$$

So the 2nd term in $(*)$ will be $< \frac{\varepsilon}{2}$ for n large.

For $n > N = \max\{N_1, N_2\}$ we have

$$(*) \leq |y| \frac{\varepsilon}{2|y|} + |x_n| \frac{\varepsilon}{2M}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Hence $x_n y_n \to xy$.

Exercise: Fix the problem about $y = 0$, & extend to general p. Hint: $x_n \to x$ if & only if x_n converges componentwise to x.

<u>Exercise:</u> Show the converse is false, i.e., give an example of x_n, y_n, x, y s.t. $x_n y_n \to xy$ but $x_n \not\to x$, $y_n \not\to y$.