16. Criteria for Convergence

**Example**

Choose any nonzero \( x \in \mathbb{R}^p \). Set \( x_n = (-1)^n x \).

The sequence \( (x_n) = (-1)^n x = (-x, x, -x, x, \ldots) \) does not converge. However, it does have convergent subsequences, e.g.,

\[
(x_{2n})_{n=1}^\infty = (x, x, x, \ldots) \rightarrow x
\]

\[
(x_{2n+1})_{n=1}^\infty = (-x, -x, -x, \ldots) \rightarrow -x.
\]
Example
Let \((r_1, r_2, \ldots)\) be a listing of all the rational nos.

Claim: There is a subsequence converging to \(\pi\).

Construction.

\[
\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \pi^2
\]

Between \(\pi^2\) & \(\pi^2 + 1\) there is some rational \(r_1\).

Between \(\pi\) & \(\pi^2\) there is some rational \(r_2\).

But to ensure we have a subsequence, we need
\[n_2 > n_1\]

Consider: Between \(\pi\) & \(\pi^2 + \frac{1}{2}\) there are \(\infty\) many rationals.
Let \(r_2\) be any one of them which also has \(n_2 > n_1\).

Between \(\pi\) & \(\pi^2 + \frac{1}{2}\) there are \(\infty\) many rationals.
\[r_4, r_5, \ldots, r_{n_2} \text{ are only fin. many; there are more.}\]

Let \(r_3\) be a rational between \(\pi\) & \(\pi^2 + \frac{1}{2}\) with \(n_3 > n_2\).

Continue in this way...
Bolzano-Weierstrass II

Every bounded sequence has a convergent subsequence.

Proof:
Assume \((x_n)\) is a bounded sequence, i.e.,
\[ \exists M \text{ s.t. } |x_n| \leq M \quad \forall n. \]

Consider
\[ S = \{x_1, x_2, x_3, \ldots\} \quad \text{set, duplicates don't matter.} \]

Case 1: \(S\) is finite. Then one value is repeated \(\infty\) many times
\[ x_{n_1} = x_{n_2} = x_{n_3} = \ldots \quad \text{for some } n_1 < n_2 < \ldots \]

Case 2: \(S\) is infinite.
Then by Bolzano-Weierstrass, \(S\) has a cluster point \(x\)
(perhaps many!).

Then \( x, x_1 \) \(\exists \) infinitely many pts of \(S\) within distance of \(r\) from \(x\).

\[ \exists n_1 \text{ s.t. } |x - x_{n_1}| < 1. \]
\[ \exists n_2 > n_1 \text{ s.t. } |x - x_{n_2}| < \frac{1}{2}. \]
\[ \exists n_3 > n_2 \text{ s.t. } \| x - x_{n_2} \| < \frac{1}{3} \quad : \]

Then \[ \| x - x_{n_k} \| < \frac{1}{k} \] so \[ x_{n_k} \to x. \]
Monotone Convergence Theorem

Assume \( X = (x_n) \subseteq \mathbb{R} \) is monotone increasing, i.e.

\[ x_1 \leq x_2 \leq x_3 \leq \ldots \]

Then: \((x_n)\) converges \(\iff\) \((x_n)\) is bounded.

Proof:
\[ \Rightarrow \] All convergent sequences are bounded (already proved).
\[ \Leftarrow \] Let \( x = \sup (x_n) \). Note \( x_n \leq x \ \forall n \).

Choose \( \varepsilon > 0 \). Since \( x - \varepsilon \) is not a lower bound for \((x_n)\),
\[ \exists N \text{ s.t. } x - \varepsilon \leq x_N. \] Note that

\[ x - \varepsilon \leq x_N \leq x_{N+1} \leq x_{N+2} \leq \ldots \leq x. \]

Hence \( |x - x_n| \leq \varepsilon \ \forall n \geq N \). So \( x_n \to x \).

Corollary: \( \sup x_n = \lim x_n \) if \((x_n)\) is bounded \& monotone increasing.
Corollary: For a monotone decreasing sequence:

\[ \lim_{n \to \infty} x_n = \inf x_n. \]

Proof: Exercise (consider \((-x_n)\)).

Definition:

\((x_n)\) is Cauchy if

\[ \forall \varepsilon > 0 \exists N > 0 \text{ st. } m, n > N \Rightarrow \|x_m - x_n\| < \varepsilon. \]

Lemma:

Convergent \(\Rightarrow\) Cauchy. [True in any normed space.]

Proof:

Suppose \(x_n \to x\). Choose \(\varepsilon > 0\) \(\exists N \text{ st. } n > N \Rightarrow \|x_n - x\| < \varepsilon/2\).

Hence if \(m, n > N\) then

\[ \|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
Example: Suppose \((x_n) \in \mathbb{R}^n\) satisfies \(\|x_n - x_m\| < \frac{1}{2^m}\). Is \((x_n)\) convergent?

Selection:

We'll show \((x_n)\) is Cauchy, hence must converge. Assume \(m > n\)

\[
\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\|
\]

\[
\leq \frac{1}{2^m} + \cdots + \frac{1}{2^n}
\]

\[
\leq \sum_{k=n}^{\infty} \frac{1}{2^k}
\]

\[
= \frac{1}{2^{n-1}}
\]

So, if we choose \(\varepsilon\) and set \(N = \log_2 \frac{1}{\varepsilon}\), then

\(\Rightarrow N \Rightarrow \|x_m - x_n\| \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^N} = \varepsilon\). \(\blacksquare\)

\((\frac{1}{n})\) is Cauchy

Ex: \((1 + \ldots + \frac{1}{n})\) is not Cauchy
Lemma: Cauchy $\implies$ Bounded

Proof:
Assume $(x_n)$ is Cauchy. Set $\varepsilon = 1$. \ \exists N \ s.t. \ \forall m \geq N \implies \|x_m - x_n\| < 1.$

In particular, $\forall m \geq N \implies \|x_m - x_N\| < 1.$

\implies \|x_m\| \leq \|x_N\| + \|x_m - x_N\| < 1 + \|x_N\|.$

So, $\|x_n\| \leq \max \{\|x_1\|, \ldots, \|x_N\|, 1 + \|x_N\|\}.$

Theorem (for $\mathbb{R}^p$):
Cauchy $\iff$ Convergent (since $\mathbb{R}^p$ is complete)

Proof:
$\implies$ Already done.

$\implies$ Assume $(x_n)$ is Cauchy. Then $(x_n)$ is bounded, hence contains a convergent subsequence $(x_{n_k})$. Let $x = \lim_{k \to \infty} x_{n_k}$.

We claim $x = \lim_{k \to \infty} x_{n_k}$.

Choose $\varepsilon > 0$. \ \exists K \ s.t. \ \forall k \geq K \implies \|x - x_k\| < \frac{\varepsilon}{3}.$

Let $k$ be s.t. $k \geq K$ and $n_k \geq N$.

In particular, $\|x - x_{n_k}\| < \frac{\varepsilon}{3}.$

So, $\|x\| = \|x - x_{n_k} + x_{n_k} - x_{n_k}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$
Assume \((x_n)\) is Cauchy. Then \((x_n)\) is bounded, hence contains a convergent subsequence \((x_{n_k})\).

Let \(x = \lim_{k \to \infty} x_{n_k}\). We claim that \(x = \lim_{n \to \infty} x_n\).

Choose \(\varepsilon > 0\).

Since \((x_n)\) is Cauchy: \(\exists N \; s.t. \; m,n > N \implies ||x_m - x_n|| < \frac{\varepsilon}{2}\).

Since \(x_{n_k} \to x\): \(\exists K \; s.t. \; k \geq K \implies ||x_{n_k} - x|| < \frac{\varepsilon}{2}\).

Let \(M > N \; \& \; M = n_k \; w.h.l. \; k \geq K\), if \(n \geq M\), then

\[ ||x - x_n|| \leq ||x - x_M|| + ||x_M - x_n|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Hence \(x_n \to x\). \(\square\)
Definition
A normed vector space \( V \) is complete if \((x_n)\) is convergent \(\iff\) \((x_n)\) is Cauchy.

Note: \(\Rightarrow\) is always true in any normed space.
\(\Leftarrow\) can fail.

Ex. True for \(\mathbb{R}^p\)
\(\ell^p\)
\(C_0\)

Ex. Fails for \(V = C'(\mathbb{R}) = \{\text{all differentiable functions on } \mathbb{R}\}\)

if the norm is \(\|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x)|\).

We'll return to this later.
Review

Visualization of $\mathbb{R}^p$ as space of digital functions

Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$. Visualize as

```
  0  1  2  $x_0$  \ldots  $x_p$
```

Each vector in $\mathbb{R}^p$ corresponds to a unique discrete function with heights $x_1, \ldots, x_p$.

The *Euclidean length* or *Euclidean norm* of $x$ is

$$
| |x| | = | |x| |_2 = \sqrt{x_1^2 + \cdots + x_p^2}.
$$

Note

$$
| |x| |^2 = \text{area under}
$$

This is not the only way to measure length.

Other, non-physical, definitions of the length of $x$ include
$\ell^1$-norm

$$\|x\|_1 = |x_1| + \ldots + |x_p| = \text{area under}$$

or

$$\|x\|_\infty = \max \{ |x_1|, \ldots, |x_p| \} = \text{max height in this picture}$$

$\|x\|_\infty$ is called the $\ell^\infty$-norm, sup-norm, or uniform norm.

**Definition**

A norm on $\mathbb{R}^p$ is a function $\|x\|$ on $\mathbb{R}^p$ satisfying

(implicitly) $\|x\|$ is a real number for each $x \in \mathbb{R}^p$

(i) $\|x\| \geq 0 \ \forall x$

(ii) $\|x\| = 0 \iff x = 0$

(iii) $\|cx\| = |c| \|x\| \ \forall c \in \mathbb{R}, x \in \mathbb{R}^p$

(iv) $\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in \mathbb{R}^p$.

The distance between $x, y \in \mathbb{R}^p$ w.r.t. this norm is the number

$$\|x-y\|.$$
Problem: How to measure distance in $\mathbb{R}^\infty$?

Example: $\|x\|_1 = |x_1| + |x_2| + \cdots = \sum_{k=1}^{\infty} |x_k|$

can be infinite!

$x = (1, 1, 1, \ldots) \Rightarrow \|x\|_1 = \infty$.

Solution

Restrict attention to subsets of $\mathbb{R}^\infty$ of vectors with finite length.

$L^1 = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \}$

$L^2 = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \|x\|_2 = \left( \sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} < \infty \}$

$L^\infty = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \|x\|_\infty = \sup \{ |x_k| \} < \infty \}$

Exercise: $L^1 \subseteq L^2 \subseteq L^\infty \subseteq \mathbb{R}^\infty$.

Exercise: $\|x\|_1$ is a norm on $L^1$.

$\|x\|_2$ is a norm on $L^2$.

$\|x\|_\infty$ is a norm on $L^\infty$.

and $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \quad \forall x \in \mathbb{R}^\infty$. 
Analog functions

Compare \( x = (x_1, \ldots, x_0) \in \mathbb{R}^n \)

to a function

\[ f: [0,1] \to \mathbb{R} \]

Both a functions, one with a discrete domain, the other with a continuous domain!

Functions are the "analogous" versions of vectors on \( \mathbb{R}^n \)!

Compare:

\( L^1 \)-norm of \( x \in \mathbb{R}^n \)

\[ \| x \|_1 = |x_1| + \cdots + |x_0| \]

area under

\( L^1 \)-norm of \( f: [0,1] \to \mathbb{R} \)

\[ \| f \|_1 = \int_0^1 |f(x)| \, dx \]

area under

Remark: We're glossing over the meaning of the integral if \( f \) is not continuous!
\( \mathbb{R}^p = \text{set of all vectors} \quad x = (x_1, \ldots, x_p) \)

\( L'_{[0,1]} = \text{set of all functions } f : [0,1] \to \mathbb{R} \)

Note \( \|x\|_1 < \infty \) for all \( x \in \mathbb{R}^p \)

\( L^1_{[0,1]} = \{ f : [0,1] \to \mathbb{R} : \|f\|_1 = \int_0^1 |f(x)| \, dx < \infty \} \)

Example: \( f(x) = x \quad f \in L^1_{[0,1]} \)

\( g(x) = 1 \quad g \in L^1_{[0,1]} \)

\( h(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0 & x = 0 \end{cases} \quad h \notin L^1_{[0,1]} \)

\( k(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0 & x = 1 \end{cases} \quad k \in L^1_{[0,1]} \)

Remark: The domain \([0,1]\) is for convenience only. You could replace it by any domain \( D \subset \mathbb{R} \) & get a set of functions \( L'(D) \). For example,

\( L'(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : \|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \} \)

\( f(x) = e^{-x^2} \quad f \in L'(\mathbb{R}) \).

\( g(x) = \sin x \quad g \notin L'(\mathbb{R}) \).
Analogues of $L^2$ & $L^\infty$ norms yield other function spaces.

$$\|x\|_2 = \left( x_1^2 + \ldots + x_p^2 \right)^{1/2}$$

$$\|f\|_2 = \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2}$$

$$L^2[0,1] = \{ f : [0,1] \to \mathbb{R} : \|f\|_2 < \infty \}$$

$$\|x\|_\infty = \max \{ |x_1|, \ldots, |x_p| \}$$

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

$$L^\infty[0,1] = \{ f : [0,1] \to \mathbb{R} : \|f\|_\infty < \infty \}$$

**Exercise:** $L^\infty[0,1] \nsubseteq L^2[0,1] \nsubseteq L'[0,1]$

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty \quad \forall f$$

**Exercise (tricky):** What about $L^\infty(\mathbb{R})$, $L^2(\mathbb{R})$, $L'(\mathbb{R})$?

**Remark:** Meaning of integral or supremum is tricky for arbitrary functions. But if we restrict to subspace of continuous functions then everything is OK using the ordinary Riemann integral.
\[ |f(x) - f_0(x)| \]

This value would be \( \|f - f_0\|_{\infty} \)

\[ |f(x) - g(x)| \leq \|f - g\|_{\infty} \leq d \quad \forall x \]

\( f \) & \( g \) are far apart in \( L^\infty \) norm.
Other possible norms 

\[ \|f\|_1 = \int |f(x)| \, dx \]

\[ L^1(\mathbb{R}) = \{ f : \int |f(x)| \, dx < \infty \} \]

\[ \|f-g\|_1 = \int |f(x)-g(x)| \, dx \]

A graph is shown with two functions, \( f \) and \( g \), to illustrate the concept of area of difference.

Area of difference < \( \varepsilon \): This \( f \) & \( g \) are close in \( L^1 \) norm but not in \( L^\infty \) norm.

\[ L^2(\mathbb{R}) \quad \|f\|_2 = \left( \int |f(x)|^2 \, dx \right)^{\frac{1}{2}} \]

Analogue of physical distance.
Summary

Typical settings in applications:

(a) Spaces of sequences of finite length \( \mathbb{R}^p \)
   
   Vectors are \( x = (x_1, \ldots, x_p) \)

(b) Spaces of sequence of infinite length – \( \ell^1, \ell^2, \ell^\infty, \mathbb{R}^\infty \)

   Vectors are \( x = (x_1, x_2, \ldots) \)

(c) Spaces of functions on a domain \( D = L^1, L^2, L^\infty \)

   Vectors are functions \( f \)

(d) Even more general settings

   E.g., functions mapping \( \mathbb{R}^p \rightarrow \mathbb{R}^2 \)

   For now, we'll concentrate on functions \( f: \mathbb{R} \rightarrow \mathbb{R} \)

   or \( f: D \rightarrow \mathbb{R} \) where \( D \subseteq \mathbb{R} \).