16. Criteria for Convergence

Example

Choose any nonzero $x \in \mathbb{R}^p$. Set $x_n = (-1)^n x$.

The sequence $(x_n) = \left((-1)^n x\right) = (-x, x, -x, x, \ldots)$

does not converge. However, it does have

convergent subsequences, e.g.,

$$(x_{2n})_{n=1}^{\infty} = (x, x, x, \ldots) \longrightarrow x$$

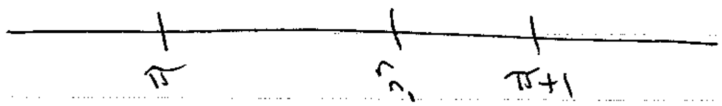
$$(x_{2n+1})_{n=1}^{\infty} = (-x, -x, -x, \ldots) \longrightarrow -x.$$

Example

Let $(r_1, r_2, \ldots)$ be a listing of all the rational nos.

Claim: There's a subsequence converging to π.

Construction.



Between π & $\pi+1$ there is some rational r_{n_1}.

Between π & $\pi+\frac{1}{2}$ " " " " r_{n_2}.

But, to ensure we have a subsequence, we need to be sure that $n_2 > n_1$.

Consider: Between π & $\pi+\frac{1}{2}$ there are ∞ many rationals.

Let r_{n_2} be any one of them which also has $n_2 > n_1$.

Between π & $\pi+\frac{1}{3}$ there are ∞ many rationals.

$r_1, \ldots, r_{n_2}$ are only fin. many – there are more.

Let r_{n_3} be a rational between π & $\pi+\frac{1}{3}$ with $n_3 > n_2$.

Continue in this way... $\square$

Bolzano - Weierstrauss II

Every bounded sequence has a convergent subsequence.

Proof:
Assume (x_n) is a bounded sequence, ie.,

$$\exists M \text{ st. } \|x_n\| \leq M \ \forall n.$$

Consider
$$S = \{x_1, x_2, x_3, \dots\}$$
set, duplicates don't matter.

Case 1: S is finite. Then one value is repeated ∞ many times
$$x_{n_1} = x_{n_2} = x_{n_3} = \cdots \quad \text{for some } n_1 < n_2 < \cdots$$

Case 2: S is infinite.
Then by Bolzano-Weierstrauss, S has a cluster pt x (perhaps many!).

Then $\forall r > 0$ $\exists$ infinitely many pts of S within a distance of r from x.

$\exists n_1$ st. $\|x - x_{n_1}\| < 1$.

$\exists n_2 > n_1$ st. $\|x - x_{n_2}\| < 1/2$

$\exists n_3 > n_2$ st. $\|x - x_{n_3}\| < 1/3$

$\vdots$

Then $\|x - x_{n_k}\| < \frac{1}{k}$ so $x_{n_k} \longrightarrow x$. ▨

Monotone Convergence Theorem

sequence of numbers

Assume $X = (x_n) \subseteq \mathbb{R}$ is monotone increasing, ie.

$$x_1 \le x_2 \le x_3 \le \cdots$$

Then: (x_n) converges $\Longleftrightarrow$ (x_n) is bounded.

Proof:

$\Rightarrow$ All convergent sequences are bounded (already proved).

$\Leftarrow$ Let $x = \sup(x_n)$. Note $x_n \le x \quad \forall n$.

Choose $\varepsilon > 0$. Since $x - \varepsilon$ is not a lower bound for (x_n),

$\exists N$ st. $x - \varepsilon \le x_N$. Note then that

$$x - \varepsilon \le x_N \le x_{N+1} \le x_{N+2} \le \cdots \quad \le x.$$

Hence $|x - x_n| \le \varepsilon \quad \forall\, n \ge N$. so $x_n \longrightarrow x$. $\blacksquare$

Corollary: $\sup x_n = \lim x_n$ if (x_n) is bounded & monotone increasing.

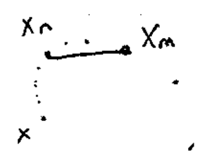
Corollary: For Monotone decreasing: bounded $\Leftrightarrow$ convergent, $\lim_n x_n = \inf_n x_n$.

Proof: Exercise (Consider $(-x_n)$).

Definition

(x_n) is CAUCHY if

$$\forall \varepsilon > 0 \ \exists N > 0 \text{ s.t. } m, n \geq N \implies \|x_m - x_n\| < \varepsilon.$$



Lemma

Convergent $\Rightarrow$ Cauchy. [True in any normed space].

Proof

Suppose $x_n \to x$. Choose ε. $\exists N$ st. $n \geq N \Rightarrow \|x_n - x\| < \varepsilon/2$.

Hence if $m, n \geq N$ then

$$\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \blacksquare$$

<u>Example</u> Suppose $(x_n) \subseteq \mathbb{R}^p$ satisfies $\|x_n - x_{n+1}\| < \frac{1}{2^n}$.
Is (x_n) convergent?

<u>Solution</u>

We'll show (x_n) is Cauchy, hence must converge. Assume $m > n$

$$
\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&< \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^n} \\
&\leq \sum_{k=n}^{\infty} \frac{1}{2^k} \\
&= \frac{1}{2^{n-1}}
\end{aligned}
$$

So, if we choose ε & set $N = \log_2 \frac{1}{\varepsilon}$, Then

$$n \geq N \Rightarrow \|x_m - x_n\| \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^N} = \varepsilon. \quad \square$$

$\left(\frac{1}{n}\right)$ is Cauchy

<u>Ex</u>: $\left(1 + \cdots + \frac{1}{n}\right)$ is not Cauchy

Lemma Cauchy $\Rightarrow$ Bounded

Proof:

Assume (x_n) is Cauchy. Set $\varepsilon = 1$. $\exists N$ s.t. $m,n \geq N \Rightarrow \|x_m - x_n\| < 1$.

In particular, $m \geq N \Rightarrow \|x_m - x_N\| < 1$.

$$\Rightarrow \|x_m\| \leq \|x_N\| + \|x_m - x_N\| < 1 + \|x_N\|.$$

So

$$\|x_n\| \leq \max\{\|x_1\|, \ldots, \|x_{N-1}\|, 1 + \|x_N\|\}. \quad \blacksquare$$

Theorem (for $\mathbb{R}^p$)

Cauchy $\Leftrightarrow$ Convergent ($\mathbb{R}^p$ is <u>complete</u>)

Proof:

$\Leftarrow$ Already done.

$\Rightarrow$ Assume (x_n) is Cauchy. Then (x_n) is bounded, hence contains a convergent subsequence (x_{n_k}). Let $x = \lim_{k\to\infty} x_{n_k}$.
We claim $x = \lim_{n\to\infty} x_n$.

$\exists N$ s.t. $m,n \geq N \Rightarrow \|x_m - x_n\| < \frac{\varepsilon}{2}$

Choose $\varepsilon > 0$. $\exists K$ s.t. $k \geq K \Rightarrow \|x - x_{n_k}\| < \frac{\varepsilon}{2}$.

Let k be s.t. $k \geq K$ & $n_k \geq N$.

In particular, $\|x - x_{n_k}\| < \frac{\varepsilon}{2}$. So, if $n \geq \max\{N, n_k\}$ then

$$\|x - x_n\| \leq \|x - x_{n_k}\| + \|x_{n_k} - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow$ Assume (x_n) is Cauchy. Then (x_n) is bounded, hence contains a convergent subsequence (x_{n_k}).

Let $x = \lim_{k \to \infty} x_{n_k}$. We claim that $x = \lim_{n \to \infty} x_n$.

Choose $\varepsilon > 0$.

Since (x_n) is Cauchy: $\exists N$ s.t. $m, n \geq N \Rightarrow \|x_m - x_n\| < \frac{\varepsilon}{2}$.

Since $x_{n_k} \to x$: $\exists K$ s.t. $k > K \Rightarrow \|x_{n_k} - x\| < \frac{\varepsilon}{2}$.

Let M be s.t. $M \geq N$ & $M = n_k$ with $k \geq K$. If $n \geq M$, then

$$\|x - x_n\| \leq \|x - x_M\| + \|x_M - x_n\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $x_n \to x$. $\blacksquare$

<u>Definition</u>

A normed vector space V is <u>complete</u> if

$$(x_n) \text{ is convergent } \iff (x_n) \text{ is Cauchy.}$$

Note: $\Rightarrow$ is always true in any normed space.
$\Leftarrow$ can fail.

<u>Ex</u>. True for $\mathbb{R}^p$
ℓ^p
c_0

<u>Ex</u>. Fails for $V = C^1(\mathbb{R}) = \{\text{all differentiable functions on } \mathbb{R}\}$

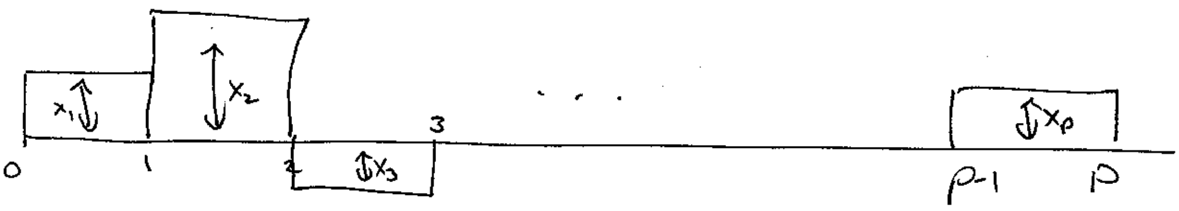
if the norm is $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$

We'll return to this later.

Review

<u>Visualization of $\mathbb{R}^p$ as space of digital functions</u>

Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$. Visualize as



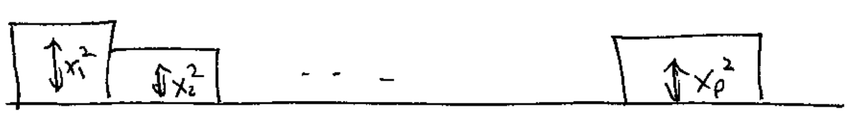
Each vector in $\mathbb{R}^p$ corresponds to a unique discrete function with heights $x_1, \ldots, x_p$.

The [redacted] <u>Euclidean length</u> or <u>Euclidean norm</u> of x is

$$\|x\| = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_p^2}.$$

Note

$\|x\|^2 =$ area under



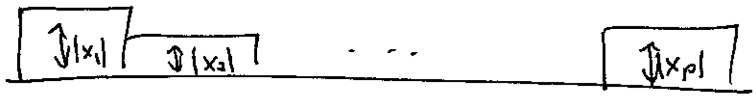
This is not the only way to measure length.

Other, <u>non-physical</u>, definitions of the length of x include

ℓ^1-norm

$\|x\|_1 = |x_1| + \cdots + |x_p| =$ area under

or



$\|x\|_\infty = \max\{|x_1|, \dots, |x_p|\}$

$=$ max height in this picture

$\|x\|_\infty$ is called the ℓ^∞-norm, sup-norm, or uniform norm.

Definition

A norm on $\mathbb{R}^p$ is a function $\|x\|$ on $\mathbb{R}^p$ satisfying

(implicitly) $\|x\|$ is a real number for each $x \in \mathbb{R}^p$

(i) $\|x\| \geq 0 \quad \forall x$

(ii) $\|x\| = 0 \iff x = 0$

(iii) $\|cx\| = |c| \, \|x\| \quad \forall c \in \mathbb{R},\ x \in \mathbb{R}^p$

(iv) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^p$.

The distance between $x, y \in \mathbb{R}^p$ w.r.t. this norm is the number

$$\|x-y\|.$$

Problem: How to measure distance in $\mathbb{R}^\infty$?

Example: $\|x\|_1 = |x_1| + |x_2| + \cdots = \sum_{k=1}^{\infty} |x_k|$

can be infinite!

$$x = (1,1,1,\dots) \Rightarrow \|x\|_1 = \infty.$$

Solution

Restrict attention to subsets of $\mathbb{R}^\infty$ of vectors with finite length.

$$\ell^1 = \{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \}$$

$$\ell^2 = \{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty : \|x\|_2 = \Big(\sum_{k=1}^{\infty} |x_k|^2 \Big)^{1/2} < \infty \}$$

$$\ell^\infty = \{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty : \|x\|_\infty = \sup |x_k| < \infty \}$$

Exercise: $\ell^1 \subsetneq \ell^2 \subsetneq \ell^\infty \subsetneq \mathbb{R}^\infty$.

Exercise: $\|x\|_1$ is a norm on ℓ^1

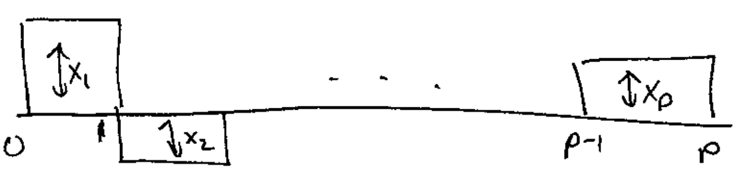
$\|x\|_2$ " " " ℓ^2

$\|x\|_\infty$ " " " ℓ^∞

and $\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \quad \forall x \in \mathbb{R}^\infty$.

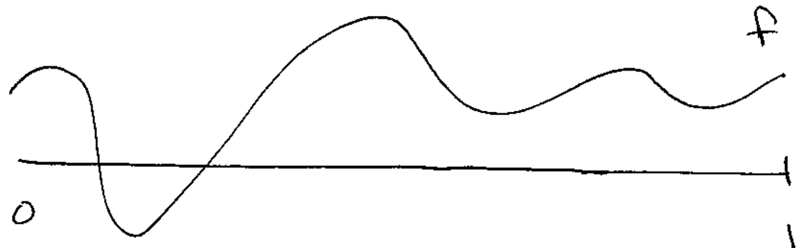
Analog functions

Compare $x = (x_1, \dots, x_p) \in \mathbb{R}^p$



to a function

$f : [0,1] \to \mathbb{R}$



Both a functions, one with a discrete domain, the other with a continuous domain!

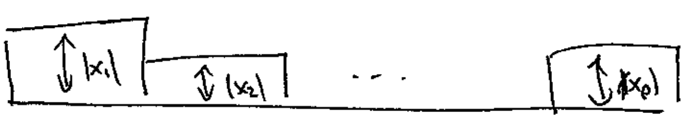
Functions are the "analog" versions of vectors in $\mathbb{R}^p$!

Compare:

ℓ^1-norm of $x \in \mathbb{R}^p$

$\|x\|_1 = |x_1| + \dots + |x_p|$

area under



L^1-norm of $f : [0,1] \to \mathbb{R}$

$\|f\|_1 = \int_0^1 |f(x)|\, dx$

area under

Remark: We're glossing over the meaning of the integral if f is not continuous!

$\mathbb{R}^p$ = set of <u>all</u> vectors $x = (x_1, \ldots, x_p)$

Note $\|x\|_1 < \infty$ for <u>all</u> $x \in \mathbb{R}^p$

$L^1[0,1]$ = set of all functions $f: [0,1] \to \mathbb{R}$ <u>for which</u> $\|f\|_1 < \infty$

$$L^1[0,1] = \left\{ f: [0,1] \to \mathbb{R} \; : \; \|f\|_1 = \int_0^1 |f(x)|\, dx < \infty \right\}$$

<u>Examples</u>: $f(x) = x \qquad f \in L^1[0,1]$

$$g(x) = 1 \qquad g \in L^1[0,1]$$

$$h(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 0 & x = 0 \end{cases} \qquad h \notin L^1[0,1]$$

$$k(x) = \begin{cases} \frac{1}{x^{1/2}}, & x > 0 \\ 0 & x = 1 \end{cases} \qquad k \in L^1[0,1]$$

<u>Remark</u>: The domain $[0,1]$ is for convenience only. You could replace it by any domain $D \subseteq \mathbb{R}$ & get a set of functions $L^1(D)$. For example,

$$L^1(\mathbb{R}) = \left\{ f: \mathbb{R} \to \mathbb{R} \; : \; \|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \right\}$$

$$f(x) = e^{-x^2} \qquad f \in L^1(\mathbb{R}).$$

$$g(x) = \sin x \qquad g \notin L^1(\mathbb{R}).$$

Analogues of ℓ^2 & ℓ^∞ norms yield other function spaces.

$$\|x\|_2 = (x_1^2 + \dots + x_p^2)^{1/2} \qquad \|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$$

$$L^2[0,1] = \{f : [0,1] \to \mathbb{R} : \|f\|_2 < \infty\}$$

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_p|\} \qquad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

$$L^\infty[0,1] = \{f : [0,1] \to \mathbb{R} : \|f\|_\infty < \infty\}$$

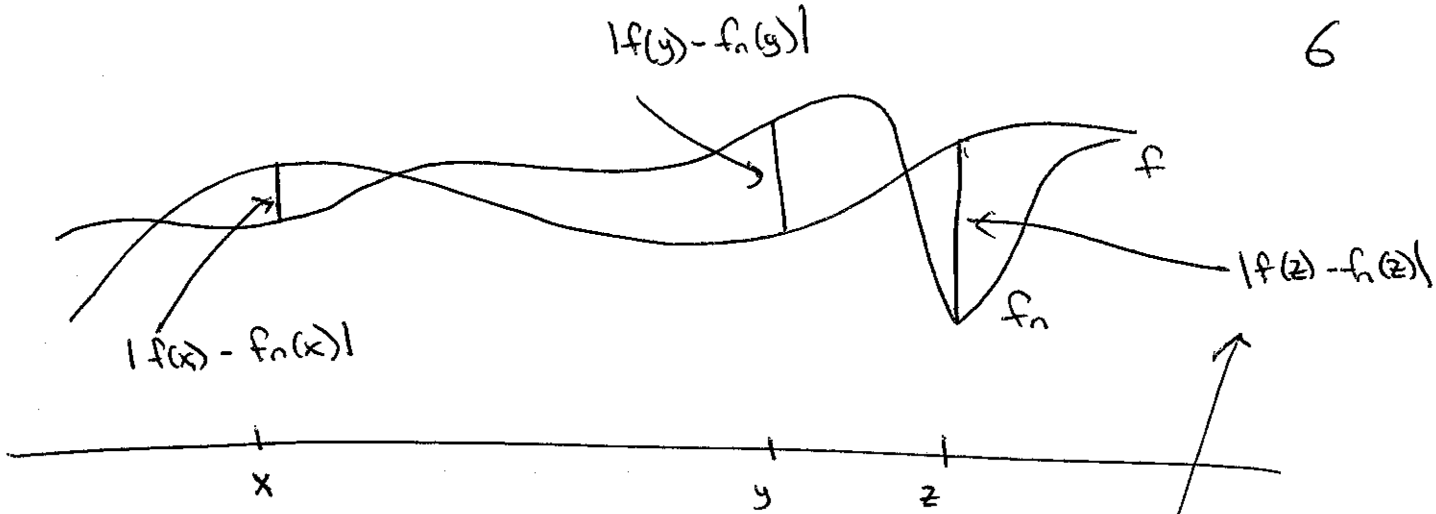
Exercise: $L^\infty[0,1] \subsetneq L^2[0,1] \subsetneq L^1[0,1]$

$$\|f\|_1 \le \|f\|_2 \le \|f\|_\infty \quad \forall f$$

Exercise (tricky): What about $L^\infty(\mathbb{R})$, $L^2(\mathbb{R})$, $L^1(\mathbb{R})$?

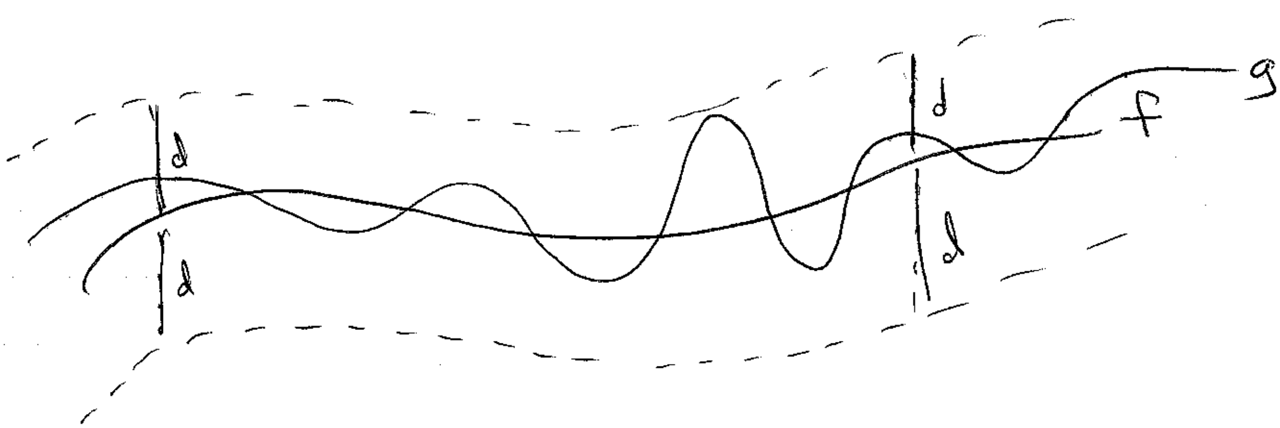
Remark: Meaning of integral or supremum is tricky for arbitrary functions. But if we restrict to subspace of continuous functions then everything is OK using the ordinary Riemann integral.

Ex.



This value would be $\|f-f_n\|_\infty$

Ex.



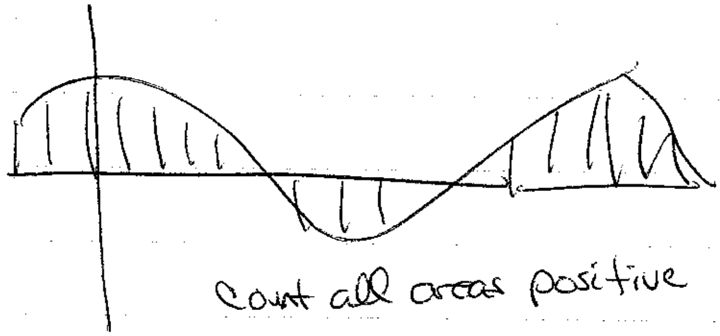
$\|f-g\|_\infty \le d$ if $|f(x)-g(x)| \le d \quad \forall x$

f & g are far apart in L^∞ norm.

<u>Other possible norms</u>

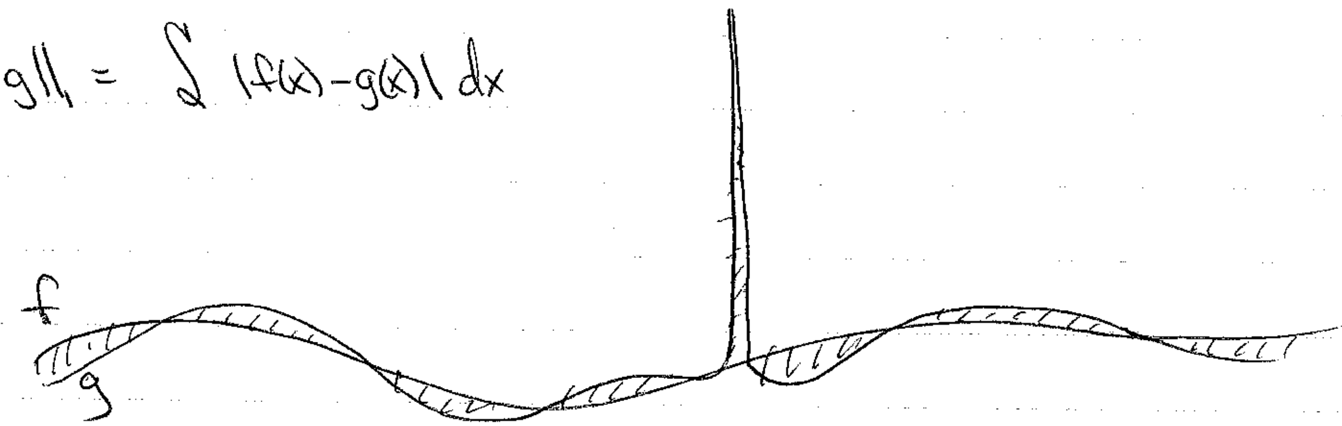
$\|f\|_1 = \int |f(x)|\,dx$

$L^1(\mathbb{R}) = \{f : \int |f(x)|\,dx < \infty\}$



count all areas positive

$\|f-g\|_1 = \int |f(x)-g(x)|\,dx$



area of difference $< \varepsilon$: this f & g are close in L^1 norm but not in L^∞ norm.

$L^2(\mathbb{R})$ $\quad \|f\|_2 = \left(\int |f(x)|^2\,dx\right)^{1/2}$ analogue of physical distance.

Summary

Typical settings in applications:

(a) Spaces of sequences of finite length $- \mathbb{R}^p$

Vectors are $x = (x_1, \dots, x_p)$

(b) Spaces of sequence of infinite length $- \ell^1, \ell^2, \ell^\infty, \mathbb{R}^\infty$

Vectors are $x = (x_1, x_2, \dots)$

(c) Spaces of functions on a domain D $- L^1, L^2, L^\infty$

Vectors are functions f

(d) Even more general settings

E.g. functions mapping $\mathbb{R}^p \longrightarrow \mathbb{R}^q$

For now, we'll concentrate on functions $f: \mathbb{R} \to \mathbb{R}$

or $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$.