**18. Limsups**

\[ (x_n) \in \mathbb{R} \]

Definition

\[
\limsup (x_n) = \limsup_{n \to \infty} x_n \quad \text{smallest cluster point is the limit}
\]

\[
\liminf (x_n) = \liminf_{n \to \infty} x_n \quad \text{largest cluster point is the liminf}
\]

\[
= \inf \{ v : \exists \text{ only finitely many } x_n > v \}
\]

\[
\limsup (x_n) = \sup \{ v : \exists \text{ only finitely many } x_n < v \}
\]

Note: \((x_n)\) not bounded above \(\Rightarrow\) no limsup (or limsup = \(\infty\))

\[\] below \(\Rightarrow\) no liminf

All bounded sequences have a liminf & limsup.

\[\] Utility: \(\liminf \leq \lim \leq \limsup \), \(\lim\) may not exist.

Lemma. Let \((x_n)\) be a bounded sequence in \(\mathbb{R}\). Then TFAE:

1. \(x^* = \limsup (x_n)\)
2. \(\forall \varepsilon > 0, \exists \text{ only finitely many } x_n > x^* + \varepsilon, \text{ but } \exists \text{ many } x_n > x^* - \varepsilon\)
3. \[x^* = \inf V_m = \inf \sup x_n\]
4. \[x^* = \lim_{m \to \infty} V_m = \lim_{m \to \infty} \sup x_n\]

5. Let \(S = \{ v : \exists (x_{n_k}) \text{ s.t. } x_n \to v \}\) Then \(x^* = \sup (S)\)
18. \text{Limsup} \quad (x_n) \in \mathbb{R} 

\text{Definition}
\limsup (x_n) = \limsup_{n \to \infty} x_n \quad \text{smallest cluster point is the liminf}
\liminf (x_n) = \sup \{v : \text{only finitely many } x_n < v\} \quad x^* = \limsup \quad \text{with only finitely many } v \quad \text{many } > v
\liminf (x_n) = \inf \{v : \text{only finitely many } x_n > v\}

\text{Note:} \quad (x_n) \text{ not bounded above } \Rightarrow \text{no limsup (or limsup = \infty)} \quad \text{not bounded below } \Rightarrow \text{no liminf}

\text{All bounded sequences have a liminf \& limsup.}

\text{Utility:} \quad \liminf \leq \lim \leq \limsup, \quad \Rightarrow \text{lim exists.}
\uparrow \text{may not exist}

\text{Lemma} \quad \text{Let } (x_n) \text{ be a bounded sequence in } \mathbb{R}. \text{ Then TFAE:}

(a) \quad x^* = \limsup (x_n)
(b) \quad \forall \varepsilon > 0, \exists \text{ only finitely many } x_n > x^* + \varepsilon, \text{ but } \exists \text{ many } x_n > x^* - \varepsilon
(c) \quad x^* = \inf \lim_{m \to \infty} \sup_{n > m} x_n
(d) \quad x^* = \lim_{m \to \infty} \sup_{n > m} x_n
(e) \quad \text{let } S = \{v : \exists (x_n) \text{ s.t. } x_n \to v\}. \text{ Then } x^* = \sup S.
Example \((x_n) = (1, -1, \frac{1}{2}, -1, \frac{1}{3}, -1, \frac{1}{4}, -1, \ldots)\)

\[V_1 = \sup_{n \geq 1} x_n = 1\]

\[V_2 = \sup_{n \geq 2} x_n = \frac{1}{2}\]

\[V_3 = \frac{1}{3}\]

\[V_4 = \frac{1}{4}\]

\[\vdots\]

\[\text{decreasing}\]

\[\inf V_n = 0 = \lim V_n \quad \text{so} \quad \lim \sup x_n = 0 \quad \lim \inf x_n = -1\]

If subsequences converging to \(-1\) & to \(0\)

largest of these is \(0\)

Note that we always have

\[V_1 \geq V_2 \geq V_3 \geq \ldots\]

But: Day could each be \(\infty\)

if some are finite, they could decrease to a finite value

or decrease forever
Proof:

(a) \implies (b) Let \( x^* = \limsup (x_n) = \inf \{v : \exists \text{ only fin. many } x_n > v \} \).

Suppose \( \varepsilon > 0 \) and there were only finitely many \( x_n > x^* - \varepsilon \).

Then \( x^* - \varepsilon \) would be one of the \( v \)'s, so the inf of all the \( v \)'s would be \( \leq x^* - \varepsilon \). But the inf is \( x^* \), so this is impossible. Here due are \( \infty \) many \( x_n > x^* - \varepsilon \).

Exercise: Here are only fin. many \( x_n > x^* + \varepsilon \).

(b) \implies (c). Suppose (b) holds. Choose \( \varepsilon > 0 \).

Then \( \exists \) only fin. many \( x_n > x^* + \varepsilon \).

Thus \( \exists N \text{ s.t. } x_n \leq x^* + \varepsilon \forall n \geq N \).

Thus \( v_N = \inf_{n \geq N} x_n \leq x^* + \varepsilon \).

So

\[ \inf_m v_m \leq v_N \leq x^* + \varepsilon. \]

This is true \( \forall \varepsilon > 0 \), so \( \inf_m v_m \leq x^* \).
Again choose \( \varepsilon > 0 \). For many \( x_n \geq x^* - \varepsilon \).

So no matter what \( m \) is, there's an \( n > m \) st. \( x_n \geq x^* - \varepsilon \), s

\[
V_m = \sup_{n \geq m} x_n \geq x^* - \varepsilon, \quad \forall m
\]

Hence

\[
\inf_m V_m \geq x^* - \varepsilon
\]

\( \inf_m V_m \) is a lower bound of \( \{ V_m \} \).

This is true \( \forall \varepsilon \) so \( \inf_m V_m = x^* = \limsup (x_n) \).

\( (c) \Rightarrow (d) \) Exercise - use the fact that the \( V_m \) are monotone decreasing.
(d) ⇒ (e) Assume (d) holds. Let \( (x_{nk}) \) be a convergent subsequence, say \( x_{n_k} \to v \). Note that \( \forall m, \)

\[
V_m = \sup_{n \geq m} x_n \geq \sup_{n_k \geq m} x_{n_k} \geq \lim_{k \to \infty} x_{n_k} = v.
\]

Hence

\[
x^* = \lim_{m \to \infty} V_m \geq v.
\]

Thus \( x^* \) is an upper bound for the set \( S \).

On the other hand, we will show \( x^* \in S \) ⇒ \( \sup(S) = x^* \).

To show \( x^* \in S \) we must show \( \exists \) subsequence \( x_{n_k} \to x^* \).

We know that \( x^* = \lim_{m \to \infty} V_m \), and \( V_m = \sup_{n \geq m} x_n \).

So:

\[
V_1 = \sup_{n \geq 1} x_n \Rightarrow \exists n_1 \text{ st. } V_1 - 1 < x_{n_1} \leq V_1
\]

\[
V_{n_1} = \sup_{n \geq n_1 + 1} x_n \Rightarrow \exists n_2 \text{ st. } V_{n_1} - \frac{1}{2} < x_{n_2} \leq V_{n_1 + 1}
\]

\[
\vdots
\]

\[
x^* = \lim_{k \to \infty} x_{n_k}
\]

Get (squeezing theorem)

\[
x^* = \lim_{n \to \infty} x_{n_k}
\]
\[ x^* = \sup_{n \geq 1} x_n, \quad \exists n, \text{ st. } x_n - \frac{1}{n} \leq x_n \leq v_n. \]

Then \[ v_2 = \sup_{n \geq 1} x_n, \quad \exists n_2 > n, \text{ st. } v_2 - \frac{1}{n_2} \leq x_{n_2} \leq v_2. \]

\[ \vdots \]

So \[ x^* = \lim_{n \to \infty} (v_n - \frac{1}{n}) \leq \lim_{n \to \infty} x_{n_k} \leq \lim_{n \to \infty} v_n = x^*. \]

Thus \[ x_{n_k} \to x^*, \text{ so } x^* \in S. \]

(e) \implies (c). Assume (e) holds. Let \[ x^* = \sup \{ s \}. \]

We want to show that \( x^* = \limsup \{ x_n \} = \inf \{ v : \text{I only fin. many } x_n < v \} \).

Choose \( \varepsilon > 0 \). If I fin. many \( x_n > x^* + \varepsilon \) then they would have a convergent subsequence \( \{ x_{n_k} \} \), say \( x_{n_k} \to v \). Then \( v \geq x^* + \varepsilon \).

But \( v \in S \), so \( \sup \{ s \} \geq v > x^* = \sup \{ s \} \), a contradiction.

Hence I only fin. many \( x_n > x^* + \varepsilon \). Hence \( x^* + \varepsilon \) is an upper bound for \( \{ v : \text{I only fin. many } x_n < v \} \).
So \( x^* + \varepsilon \geq \limsup (x_n) \). True for all \( \varepsilon \), so \( x^* \geq \limsup (x_n) \).

On the other hand, \( x^* = \sup (S) \), so \( \exists \varepsilon > 0 \) s.t.

\[ V \ni x^* - \varepsilon \frac{1}{2}. \] Hence \( \exists \{x_{n_k}\} \) s.t. \( x_{n_k} \rightarrow V > x^* - \varepsilon > x^* - \varepsilon, \)

Therefore \( \exists \) many \( x_{n_k} > x^* - \varepsilon. \)

\[ \limsup (x_n) = \inf \{v : \exists \text{ only fin. many } x_n > v\} \geq x^* - \varepsilon. \]

True for all \( \varepsilon \), so \( \limsup (x_n) \geq x^* \).

Properties / Exercise

(a) \( \liminf (x_n) \leq \limsup (x_n) \)

(b) \( \limsup (x_n + y_n) \leq \limsup (x_n) + \limsup (y_n) \) \[ \text{But NOT for } \liminf \]

\[ \liminf (x_n + y_n) \geq \liminf (x_n) + \liminf (y_n) \]

(c) \( x_n \leq y_n \ \forall n \Rightarrow \liminf (x_n) \leq \liminf (y_n) \)

\[ \limsup (x_n) \leq \limsup (y_n) \]

\[ \limsup (x_n + y_n) = \inf \left[ \sup_{n \geq m} (x_n + y_n) \right] \leq \limsup x_n + \limsup y_n \]
Example \( \liminf (x_n+y_n) \neq \liminf x_n + \liminf y_n \)

\( (x_n) = (1, -1, 1, -1, \ldots) \quad \liminf x_n = -1 \)

\( (y_n) = (-1, 1, -1, 1, \ldots) \quad \liminf y_n = -1 \)

\( (x_n+y_n) = (0, 0, \ldots) \quad \liminf (x_n+y_n) = 0 \)
MATH 4318

HOMEWORK #2: SOLUTIONS

DUE: April 21, 1999

Work the following problems and hand in your solutions. A subset of these will be selected for grading.

1. (a) Use the definition

\[ \limsup_{n \to \infty} x_n = \inf \sup_{m \geq m} x_n \]

to prove carefully that

\[ \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n. \]

Hint: Give names to things, e.g., define \( u_m = \sup_{n \geq m} a_n \), \( v_m = \sup_{n \geq m} b_n \), and \( w_m = \sup_{n \geq m} (a_n + b_n) \).

Solution

In addition to the names above, define

\[ u = \inf_{m} u_m, \quad v = \inf_{m} v_m, \quad w = \inf_{m} w_m. \]

We want to show that \( w \leq u + v \).

We proved in the section on suprema that

\[ w_m = \sup_{n \geq m} (a_n + b_n) \leq \sup_{n \geq m} a_n + \sup_{n \geq m} b_n = u_m + v_m. \]

Further, \( w \) is the inf of all the \( w_m \), so we know that \( w \leq w_m \) for every \( m \). Hence \( w \leq u_m + v_m \) for every \( m \).

Let \( \varepsilon > 0 \) be given. Since the inf of the \( u_m \) is \( u \), there exists some \( k \) such that \( u_k < u + \varepsilon \). Since the \( u_m \) are decreasing, we conclude that \( u_m \leq u_k < u + \varepsilon \) for all \( m \geq k \). Similarly, there exists some \( \ell \) such that \( v_m \leq v_\ell < v + \varepsilon \) for all \( m \geq \ell \). Let \( j = \max \{k, \ell\} \). Then for \( m \geq j \) we have BOTH \( u_m < u + \varepsilon \) and \( v_m < v + \varepsilon \). Therefore, for \( m \geq j \),

\[ w \leq u_m + v_m < (u + \varepsilon) + (v + \varepsilon) = u + v + 2\varepsilon. \]

Since this is true for EVERY \( \varepsilon > 0 \), we conclude that \( w \leq u + v \).  \( \Box \)

(b) Either prove that the inequality in part (a) is an equality, or find a counterexample.

Solution

Let \( (a_n) = (1, -1, 1, -1, 1, -1, \ldots) \) and \( (b_n) = (-1, 1, -1, 1, -1, \ldots) \). Then \( a_n + b_n = (0, 0, 0, \ldots) \), so

\[ \limsup_{n \to \infty} (a_n + b_n) = 0 < 2 = 1 + 1 = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n. \]
Theorem
Let \((x_n) \subseteq \mathbb{R}\) be bounded. Then:

\[(x_n) \text{ converges} \iff \liminf x_n = \limsup x_n.\]

In this case, this is the limit.

Proof
\(\Rightarrow\). Assume \(x_n \to x\). Choose \(\varepsilon > 0\). \(\exists N\) st.

\[n > N \Rightarrow x - \varepsilon \leq x_n \leq x + \varepsilon.\]

Hence \(x - \varepsilon = \liminf (x - \varepsilon) \leq \liminf (x_n) \leq \limsup (x_n) \leq \limsup (x + \varepsilon) = x + \varepsilon.\)

True for all \(\varepsilon\), so \(x \leq \liminf (x_n) \leq \limsup (x_n) \leq x.\)

\(\Leftarrow\). Assume \(\liminf x_n = \limsup x_n = x\). Choose \(\varepsilon > 0\).

Then \(\exists N\) only finitely many \(x_n > x^* + \varepsilon\). Hence \(\exists N\), st.

\[n > N \Rightarrow x_n \leq x^* + \varepsilon.\]

Similarly \(\exists N\) st. \(n > N \Rightarrow x_n \geq x - \varepsilon\) (look at \(\liminf\)).

\(\therefore n \geq \max\{N_1, N_2\} \Rightarrow |x - x_n| \leq \varepsilon \Rightarrow x_n \to x.\)
Infinite limits

Suppose \( (x_n) \) is unbounded. Given \( R > 0 \), if there were only finitely many \( x_n > R \) then \( (x_n) \) would be bounded. Hence \( \exists \) many \( x_n > R \). Therefore, no matter what \( m \) is, \( \exists n \geq m \) s.t. \( x_n > R \). Hence

\[
V_m = \sup_{n \geq m} x_n > R, \quad \text{all } m
\]

so

\[
\limsup_{n \to \infty} x_n = \inf_{m} V_m \geq R.
\]

But \( R \) is arbitrary, so we write

\[
\limsup_{n \to \infty} x_n = \infty.
\]

Infinite limits

Given \( (x_n) \), we say \( (x_n) \) diverges to \( \infty \) if

\[
\forall R > 0 \exists N > 0 \text{ s.t. } n \geq N \Rightarrow x_n > R.
\]

In this case we write \( \lim_{n \to \infty} x_n = \infty \)

Note that \( x_n \) is not "converging to \( \infty \)! We do not get \( |x_n - \infty| < \varepsilon \)."