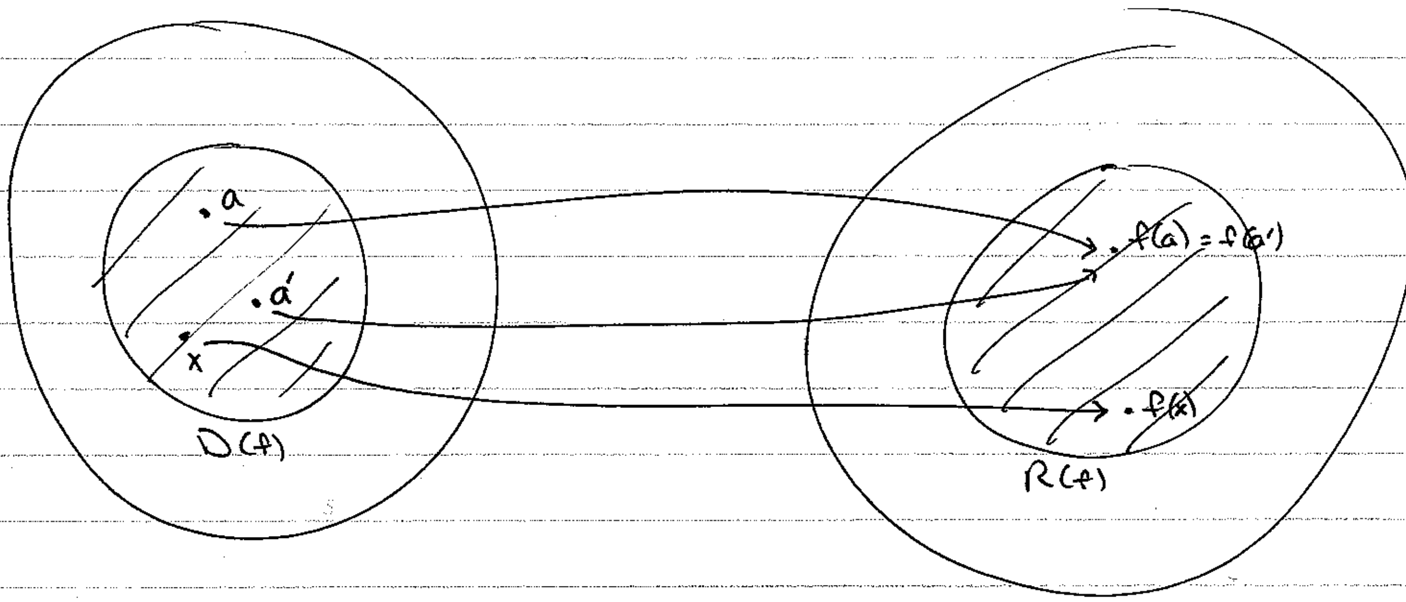
2. Functions

Informal Definition

A function f from a set A to a set B is a rule that assigns to each x in a fixed subset $D(f) \subseteq A$ a unique element $f(x) \in B$



Domain: $D(f) = \{ a \in A : \text{there is some } b \in B \text{ s.t. } b = f(a) \}$

Range: $R(f) = \{ f(a) : a \in A \}$

$= \{ b \in B : b = f(a) \text{ for some } a \in A \}$

Sometimes write $x \mapsto f(x)$

"maps to"

Definition

If $D(f) = A$ we say f maps A into B and write

$$f : A \to B. \quad \Longleftarrow \text{ Note: This notation requires domain} = A$$

If $R(f) = B$ we say f is surjective or onto.

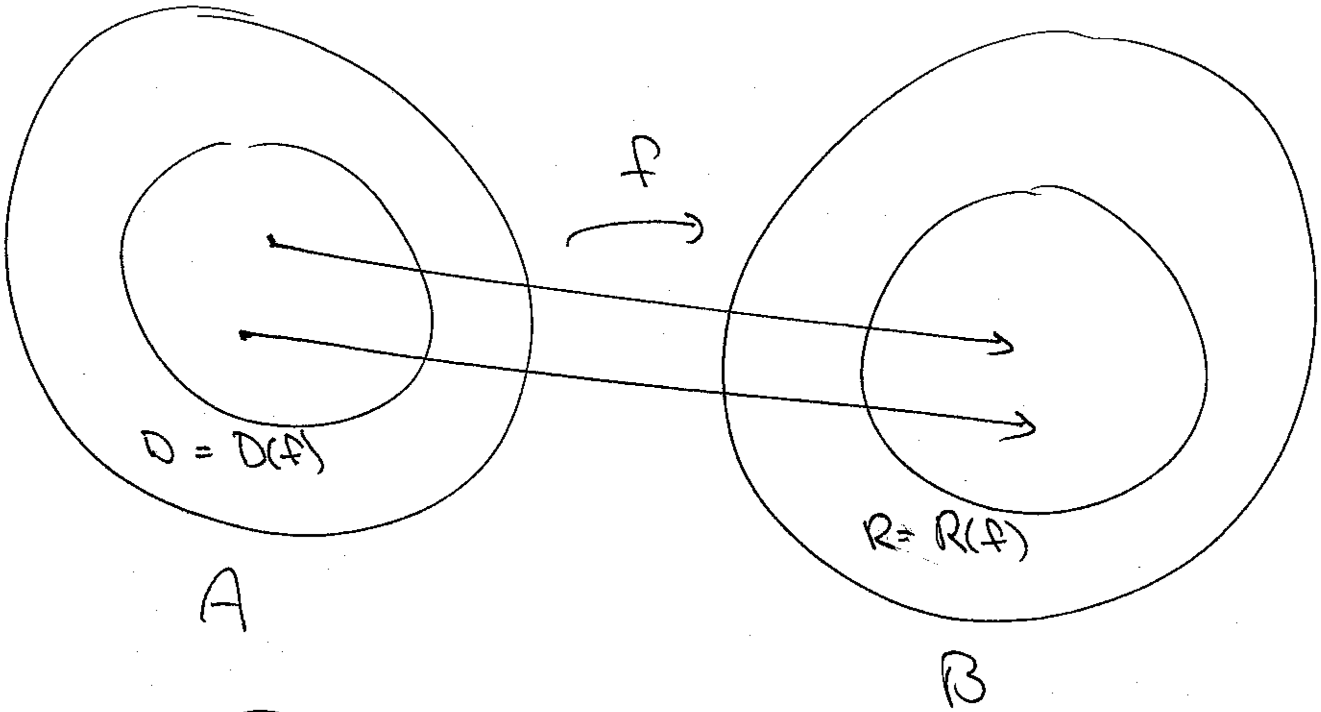
If $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$ then we say f is injective or 1-1.

Equivalent and more convenient:

f is injective if $\quad f(a_1) = f(a_2) \implies a_1 = a_2$.

f is bijective, or a 1-1 correspondence, if it is both injective and surjective.

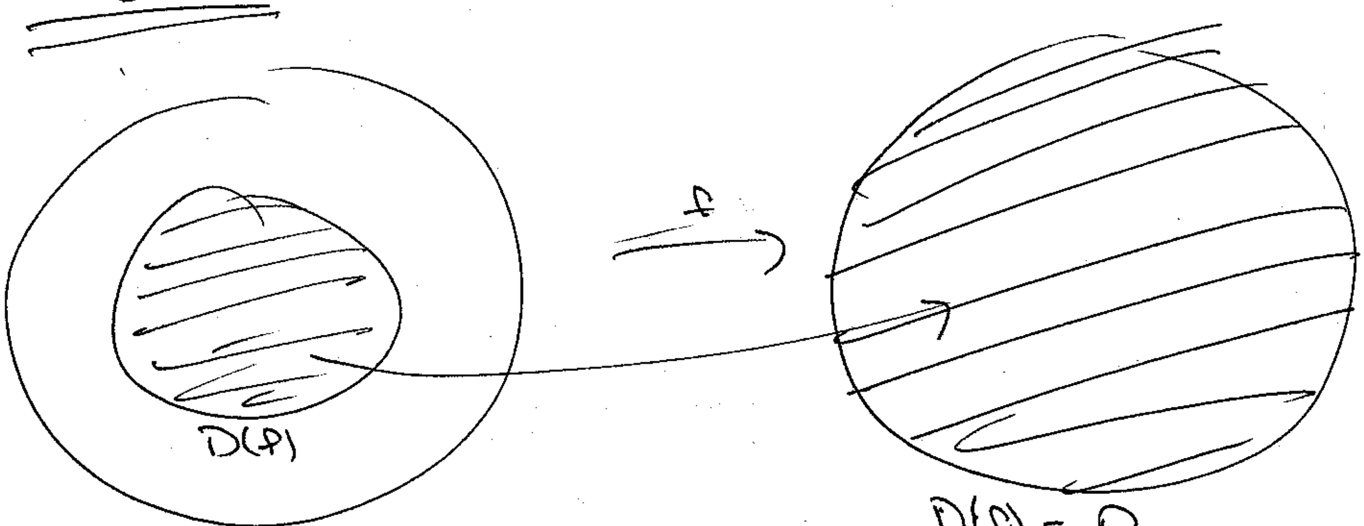
Injective (1-1)



Different points go different places.

OR: If 2 points go to the same place, then they had to be the same point to begin with.

Surjective (Onto)



$R(f) = B$

The range is everything.

Example $f(x) = \sqrt{x}$ $\quad A = B = \mathbb{R}$ $\quad$ Injective
Not surjective.

If we change B to $B = [0, \infty)$ then f is surjective.

Example Let

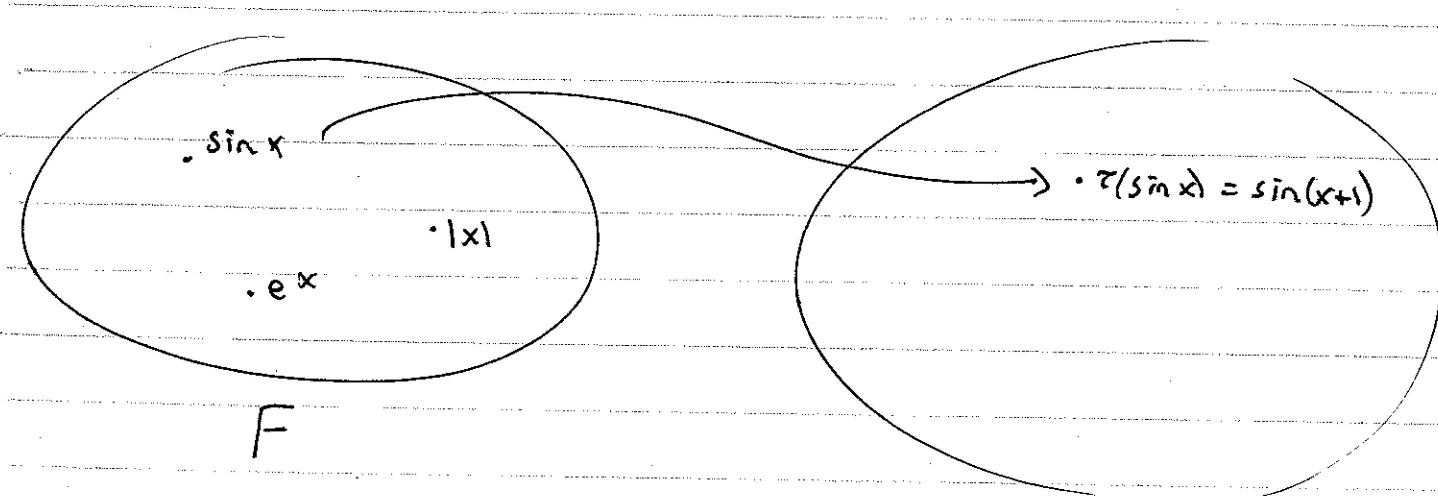
$$F = \{ f : f \text{ is a function from } \mathbb{R} \text{ into } \mathbb{R} \} \quad (f : \mathbb{R} \to \mathbb{R})$$

$f(x) = \sin x \quad g(x) = x^2 \quad h(x) = |x|$ are all elements of F

Define $\tau : F \to F$ as follows.

For each $f \in F$, $\tau(f)$ is a new function $\tau(f) : \mathbb{R} \to \mathbb{R}$, defined by

$$\big(\tau(f)\big)(x) = f(x+1)$$



<u>EXERCISE:</u> Show that τ is a bijection.

<u>Injective</u> Suppose $\tau(f) = \tau(g)$ for some $f, g \in F$.

This means that $\tau(f)$ & $\tau(g)$ are the <u>same</u> <u>function</u>, i.e., they define the <u>same</u> rule, so

$$(\tau(f))(x) = (\tau(g))(x) \quad \text{for all } x \in \mathbb{R}.$$

Hence $\quad f(x+1) = g(x+1) \quad$ for all $x \in \mathbb{R}$.

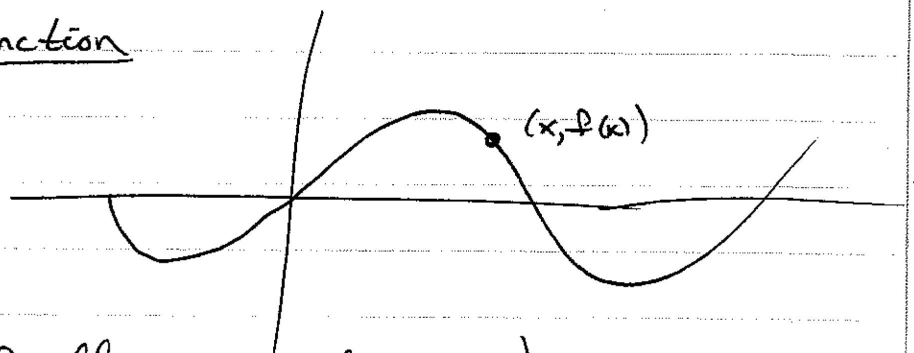
But this implies $f(t) = g(t)$ for every $t \in \mathbb{R}$,

so f & g define the same rule, hence are the same function, i.e., $f = g$.

Thus τ is injective.

Precise definition of a function

Motivation from graphs:



The graph is the set of all pairs $(x, f(x))$.

If you know f, you know the graph & conversely.

For the arbitrary setting, we define the function to be the graph.

Definition

Let A, B be sets. A function f from A to B is any set of ordered pairs $f \subseteq A \times B$ which has the property that

$$(a,b) \in f \ \& \ (a,b') \in f \implies b = b'$$

Thus, each a is associated with a unique b, called the image of a & written

$$b = f(a).$$

$$D(f) = \{ a \in A : (a,b) \in f \text{ for some } b \in B \}$$

$$R(f) = \{ b \in B : (a,b) \in f \text{ for some } a \in A \}.$$

Example: $A = B = \mathbb{R}$

$$f = \{ (x, x^{1/2}) : x \geq 0 \}$$

is the precise meaning of saying "f is the function $f(x) = \sqrt{x}$"

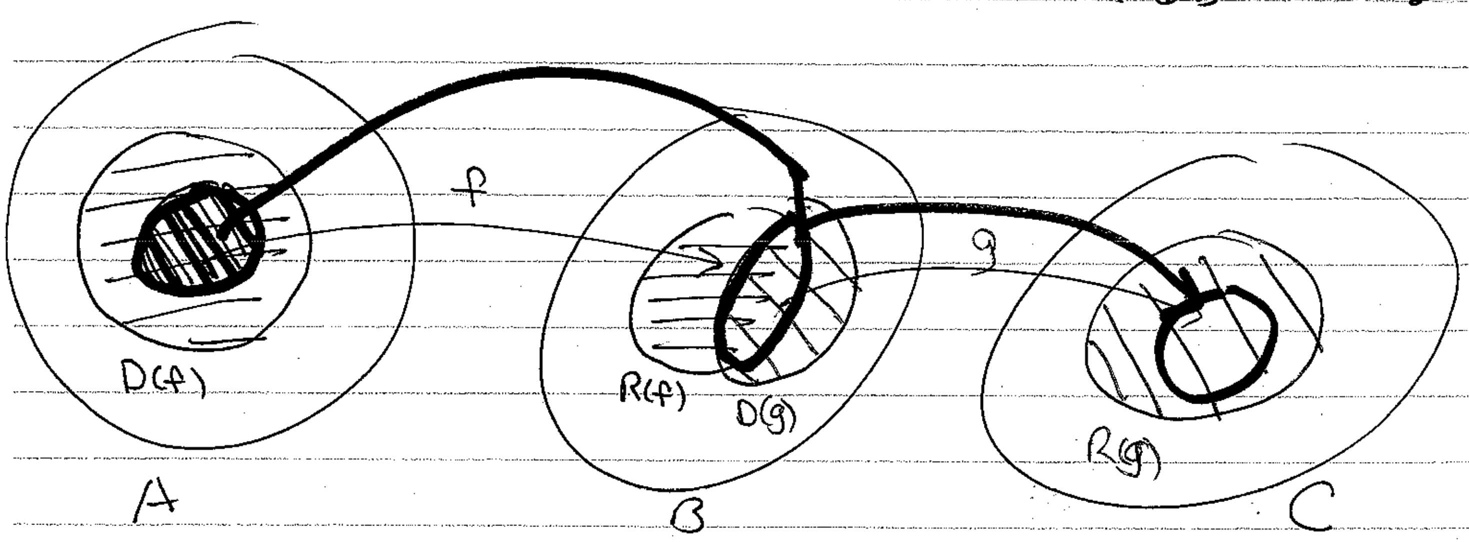
Definition Composition

If f is a function from A to B
& g " " " " B to C

then $g \circ f$ is the function from A to C defined by the rule

$$(g \circ f)(x) = g(f(x))$$

for all x's for which this makes sense



$D(g \circ f) = \{ x \in D(f) : f(x) \in D(g) \}$

$R(g \circ f) = \{ g(f(x)) : x \in D(g \circ f) \}$

<u>Example</u> Let $F = \{ \text{functions } f : f : \mathbb{R} \to \mathbb{R} \}$

Define for <u>each</u> $a \in \mathbb{R}$ a function $\tau_a : F \to F$ by

$$\big(\tau_a(f)\big)(x) = f(x+a)$$

What is $\tau_a \circ \tau_b$?

Consider any $f \in F$. $(\tau_a \circ \tau_b)(f)$ is the function

$$\Big((\tau_a \circ \tau_b)(f) \Big)(x) = \Big(\tau_a(\tau_b(f)) \Big)(x)$$

$$= \big(\tau_b(f)\big)(x+a)$$

$$= f\big((x+a) + b \big)$$

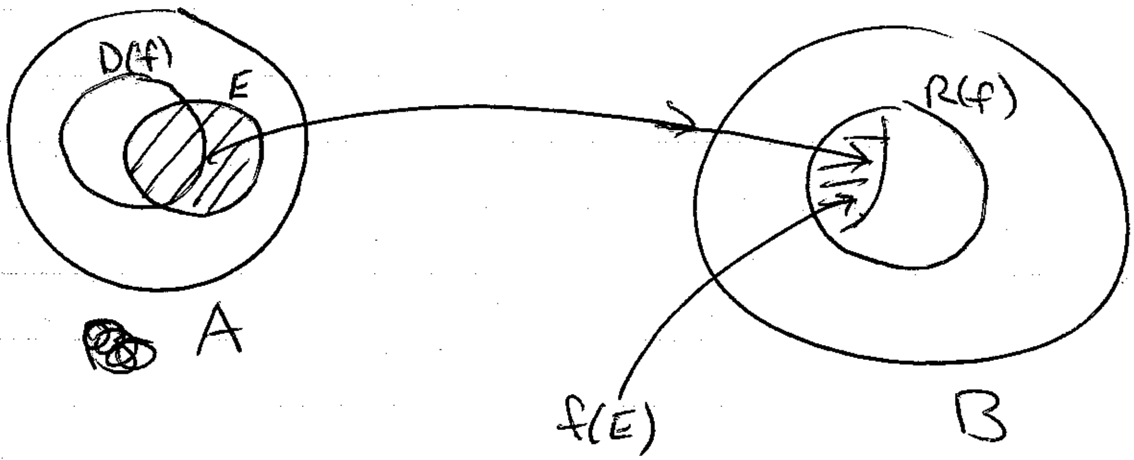
$$= f\big(x + (a+b) \big)$$

$$= \big(\tau_{a+b}(f) \big)(x)$$

Hence $(\tau_a \circ \tau_b)(f) = \tau_{a+b}(f)$. Since this is true for <u>each</u> f in the domain, $\tau_a \circ \tau_b$ & τ_{a+b} have the same rule, so are the same function, i.e. $\tau_a \circ \tau_b = \tau_{a+b}$.

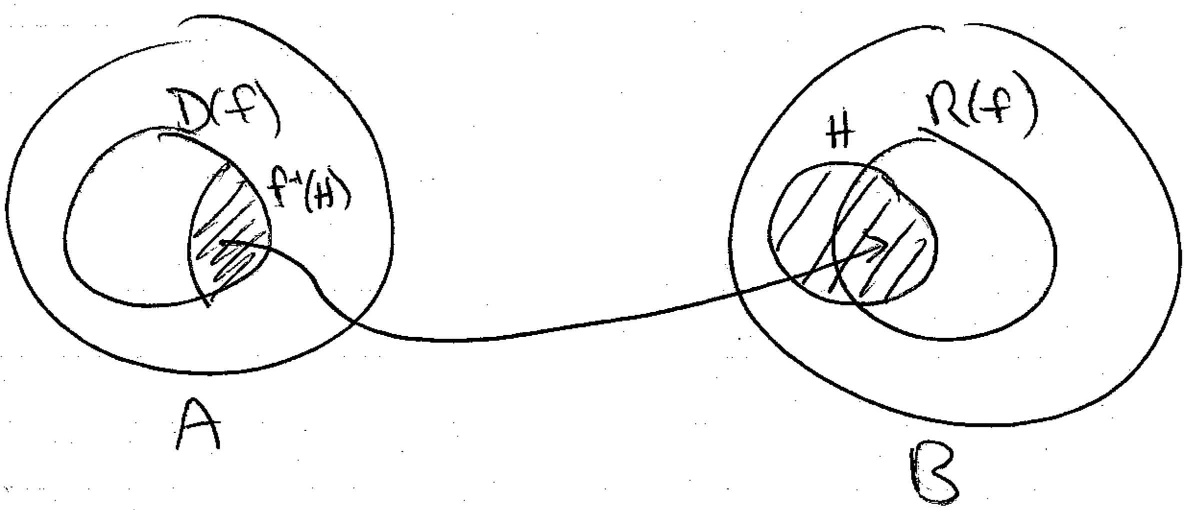
Direct Image The *direct image* of $E \subset A$ is

$$f(E) = \{ f(x) : x \in E \cap D(f) \}$$



Inverse image The *inverse image* of $H \subset B$ is

$$f^{-1}(H) = \{ x \in D(f) : f(x) \in H \}$$



Note: Does NOT require f to be injective, is NOT the same as inverse function.

Example Length of a vector.

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) \ : \ x_1, \dots, x_n \in \mathbb{R} \}$$

The length of $x \in \mathbb{R}^n$ is

$$\ell(x) = \|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$

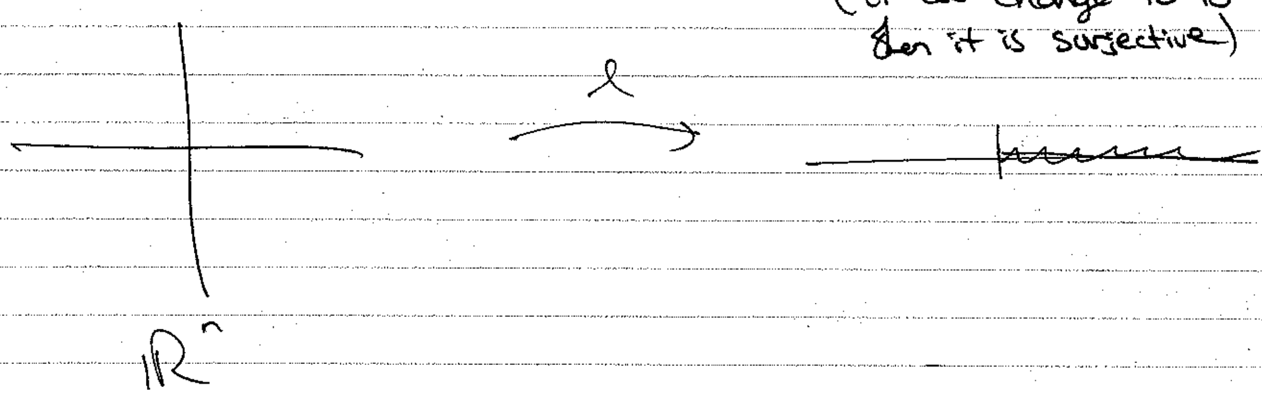
physical length or Euclidean length

Note ℓ is a function:

$$\ell : \mathbb{R}^n \longrightarrow \mathbb{R}$$

ℓ is not injective
not surjective

(but if we change to $B = [0, \infty)$
then it is surjective)



$D(\ell) = \mathbb{R}^n$

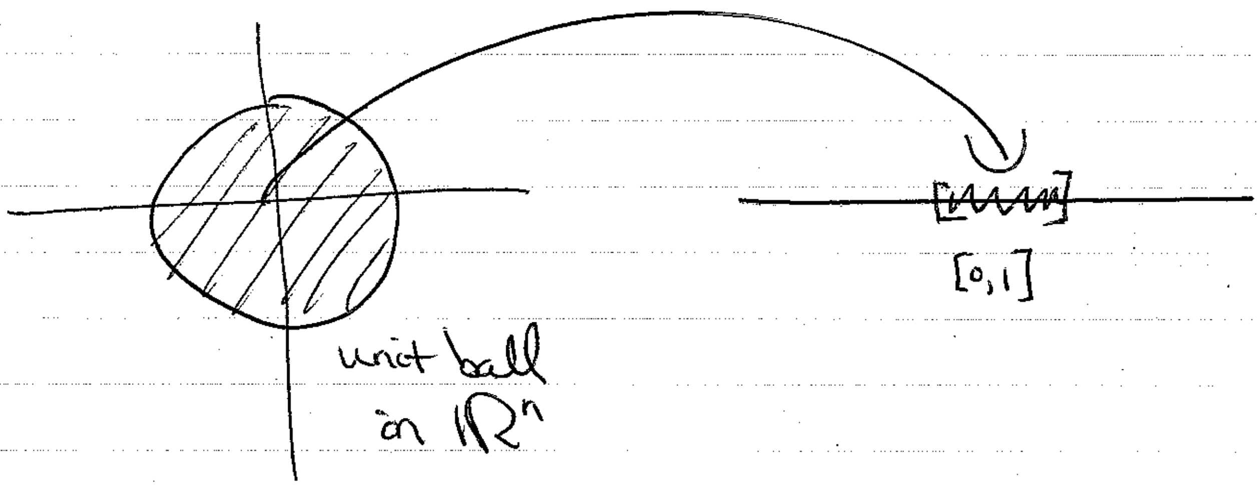
$R(\ell) = [0, \infty)$

ℓ has special properties:

study other functions with these properties later

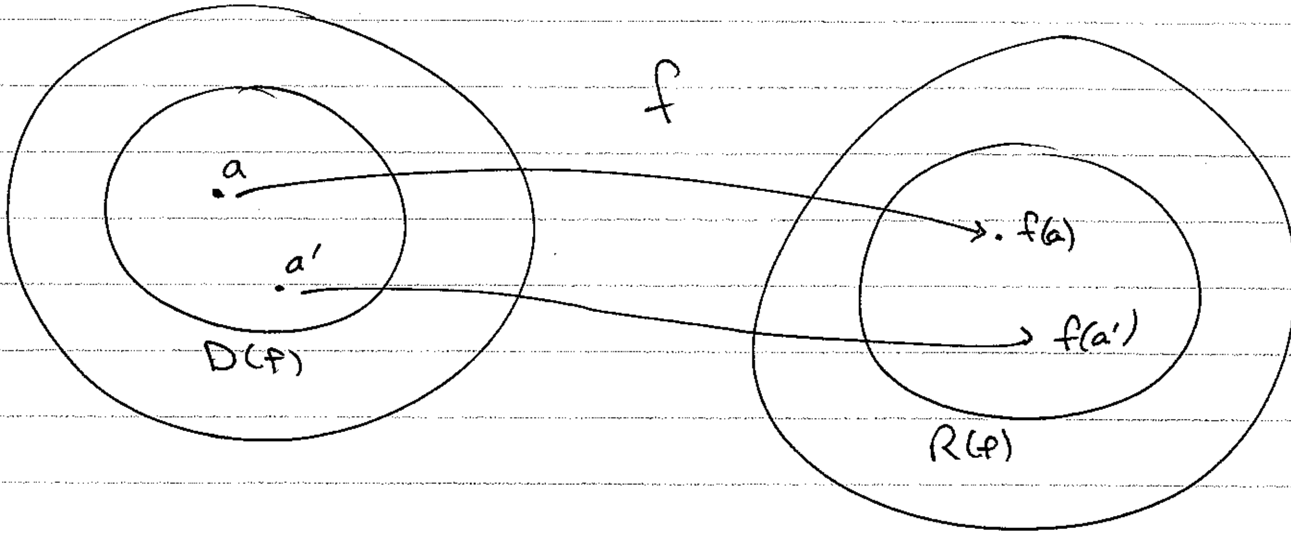
1. $\|x\| \geq 0 \ \forall x \in \mathbb{R}^n$
2. $\|x\| = 0 \iff x = 0$
3. $\| cx \| = |c| \, \|x\|$ for any scalar c
4. $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality).

$$\ell^{-1}([0,1]) = \{x \in \mathbb{R}^n : 0 \le \|x\| \le 1\}$$



Inverse Function

Assume f is injective.



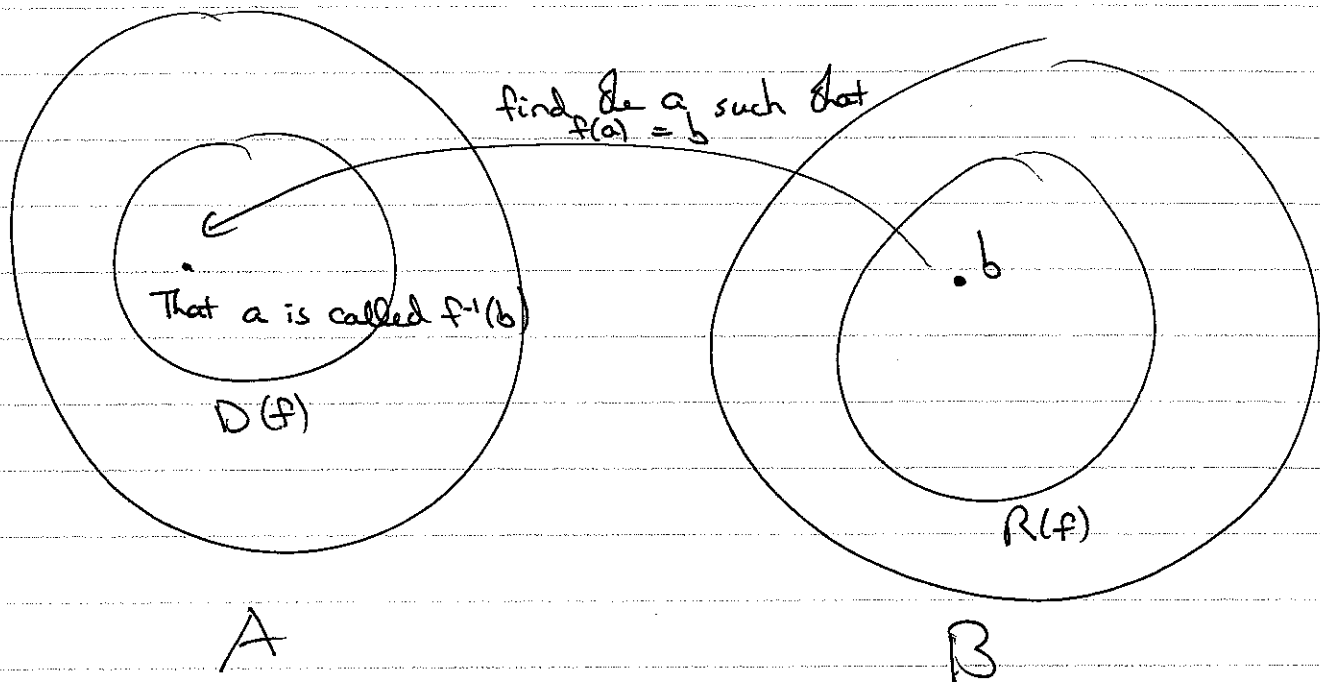
Each a is associated with a unique $f(a)$ in the sense that no other $a' \in D(f)$ has that same image

Thus each $b \in R(f)$ equals some $\overset{\text{unique}}{f(a)}$.

Each $b \in R(f)$ has a unique $a \in D(f)$ s.t. $b = f(a)$.

Define f^{-1} from B to A by

$$f^{-1}(b) = a, \quad \left(\text{the unique } a \text{ s.t. } b = f(a)\right)$$



By def: $f^{-1}\left(f(a)\right) = a$

Because a is the unique element s.t. $f(a) = f(a)$

Also: $f\left(f^{-1}(b)\right) = b$ because $a = f^{-1}(b)$ is the unique element s.t. $f(a) = b$

<u>Exercise:</u> $\tau_a : F \to F$ inverse function is $\tau_a^{-1} = \tau_{-a}$