

21. Linear Functions

A function $f: \mathbb{R}^p \rightarrow \mathbb{R}^q$ is linear if

$$\forall x, y \in \mathbb{R}^p, \forall a, b \in \mathbb{R}, f(ax+by) = af(x) + bf(y).$$

$$\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) = \{ f: \mathbb{R}^p \rightarrow \mathbb{R}^q : f \text{ is linear} \}$$

Suppose f is linear. Define

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^p$$

Set

$$f(e_1) = a_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{q1} \end{bmatrix}, \dots, f(e_p) = a_p = \begin{bmatrix} a_{1p} \\ \vdots \\ a_{qp} \end{bmatrix} \in \mathbb{R}^q.$$

Define

$$A = \begin{bmatrix} | & & | \\ a_1 & \dots & a_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \dots & a_{qp} \end{bmatrix}$$

$q \times p$ matrix.

$$\text{Let } x = (x_1, \dots, x_p) = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^p.$$

Then

$$x = x_1 e_1 + \dots + x_p e_p$$

So

$$f(x) = x_1 f(e_1) + \dots + x_p f(e_p)$$

$$= x_1 a_1 + \dots + x_p a_p$$

$$= \begin{bmatrix} a_{11}x_1 + \dots + a_{1p}x_p \\ \vdots \\ a_{q1}x_1 + \dots + a_{qp}x_p \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \dots & a_{qp} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

$$= Ax.$$

ALL LINEAR FUNCTIONS ARE GIVEN BY MATRICES.

Define $K = \max_{i,j} |a_{ij}|$

Then

$$\begin{aligned} \|f(x)\| &= \left(|a_{11}x_1 + \dots + a_{1p}x_p|^2 + \dots + |a_{q1}x_1 + \dots + a_{qp}x_p|^2 \right)^{1/2} \\ &\leq \left((|a_{11}| |x_1| + \dots + |a_{1p}| |x_p|)^2 + \dots + (|a_{q1}| |x_1| + \dots + |a_{qp}| |x_p|)^2 \right)^{1/2} \\ &\leq \left((K|x_1| + \dots + K|x_p|)^2 + \dots + (K|x_1| + \dots + K|x_p|)^2 \right)^{1/2} \end{aligned}$$

↖ q times ↗

$$= \left(q K^2 (|x_1| + \dots + |x_p|)^2 \right)^{1/2}$$

$$= q^{1/2} K \sum_{k=1}^p |x_k| \cdot 1$$

$$\leq q^{1/2} K \left(\sum_{k=1}^p |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^p 1^2 \right)^{1/2} \quad \text{Cauchy-Schwarz}$$

$$= q^{1/2} K \|x\| p^{1/2}$$

$$= (pq)^{1/2} K \|x\|.$$

So:

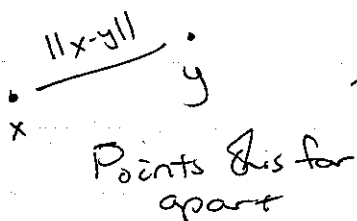
↙ f linear ↘

$$\|f(x) - f(y)\| = \|f(x-y)\| \leq (pq)^{1/2} K \|x-y\|$$

Definition

A function f is Lipschitz if $\exists K > 0$ s.t.

$$\forall x, y \in D(f), \quad \|f(x) - f(y)\| \leq K \|x - y\|.$$



$$\frac{f(x) \quad f(y)}{\|f(x) - f(y)\|} \leq K \|x - y\|$$

map to points
no more than K times
farther apart

If ~~and~~ $K < 1$ then we say f is contractive.

(In this case, $f(x)$ & $f(y)$ are always closer together than x & y were).

Note: We've shown that all linear functions are Lipschitz.

Example: $f(x) = |x|$ is Lipschitz but NOT linear.

(e.g., $f(-x) \neq -f(x)$)

Theorem

A Lipschitz function is continuous at all points in its domain.

Proof:

Suppose $x_n, a \in D(f)$ & $x_n \rightarrow a$. Then

$$\|f(x_n) - f(a)\| \leq K \|x_n - a\| \rightarrow 0.$$

Hence $f(x_n) \rightarrow f(a)$ so f is continuous at a . \square

Corollary

All linear functions are continuous.

Definition

$f: \mathbb{R}^p \rightarrow \mathbb{R}^q$ is Hölder continuous with exponent $\alpha > 0$ if

$\exists K > 0$ st.

$$\forall x, y \in \mathbb{R}^p, \quad \|f(x) - f(y)\| \leq K \|x - y\|^\alpha.$$