<u>Section 22</u>

Let f be a function from $\mathbb{R}^p$ to $\mathbb{R}^q$ with domain $D(f) \subseteq \mathbb{R}^p$ & range $R(f) \subseteq \mathbb{R}^q$.

<u>Global continuity Theorem</u>

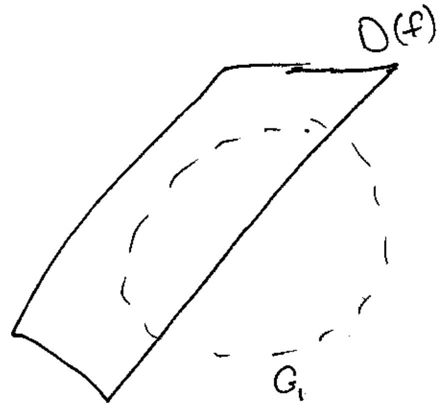
The following statements are equivalent.

(a) f is continuous at every point $a \in D(f)$ [we write "f is continuous on $D(f)$" in this case].

(b) If $G \subseteq \mathbb{R}^q$ is open, then

$$f^{-1}(G) = G_1 \cap D(f)$$

for some open $G_1 \subseteq \mathbb{R}^p$.



(c) If $H \subseteq \mathbb{R}^q$ is closed, then

$$f^{-1}(H) = H_1 \cap D(f)$$

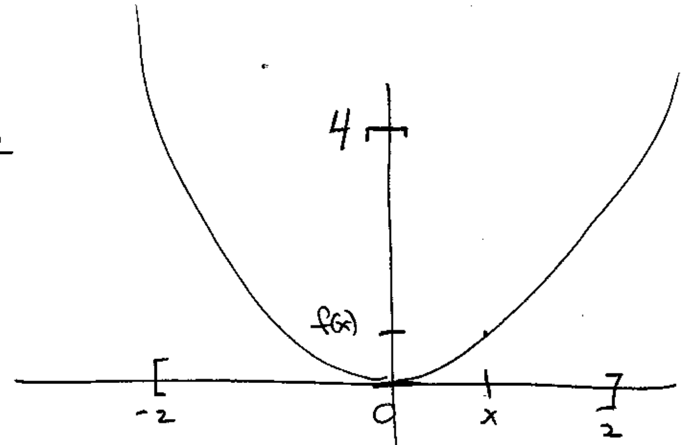
for some closed $H_1 \subseteq \mathbb{R}^p$.

<u>Remark:</u> <u>If</u> $D(f) = \mathbb{R}^p$ then we can rewrite (b) & (c) as

(b′) If $G \subseteq \mathbb{R}^q$ is open, then
$f^{-1}(G)$ is open in $\mathbb{R}^p$.

(c′) If $H \subseteq \mathbb{R}^q$ is closed, then
$f^{-1}(H)$ is closed in $\mathbb{R}^p$.

<u>But</u>, if $D(f) \neq \mathbb{R}^p$ then $f^{-1}(G)$ need not be open.
In that case, it only has to be true that $f^{-1}(G) = G_1 \cap D(f)$
where $\underline{G_1}$ is some open set.



Example $p = q = 1$ $\quad f(x) = x^2$

Note $D(f) = \mathbb{R}$.

If $G = (-1, 4)$ then $f^{-1}(G) = (-2,2)$ is open.

If $H = [-1, 4]$ then $f^{-1}(H) = [-2,2]$ is closed.

Ex: Same except restrict to $D(f) = [0, \infty)$

Then $f^{-1}(G) = [0, 2)$ is not open

$$= \underbrace{(-2,2)}_{G_1 \text{ open}} \cap D(f)$$

Proof of the Theorem

<u>(a) $\Rightarrow$ (b)</u> Suppose f is continuous on $D(f)$.

Let $G \subseteq \mathbb{R}^q$ be any open set. Recall that

$$f^{-1}(G) = \{ x \in D(f) : f(x) \in G \}.$$

Let $a \in f^{-1}(G)$. Then $f(a) \in G$, & G is open so it is a neighborhood of $f(a)$. By def. of continuity, $\exists$ neighborhood U_a of a such that $f(U_a) \subseteq G$. By def. of neighborhood, $\exists$ open set O_a such that

$$a \in O_a \subseteq U_a.$$

Define

$$G_1 = \bigcup_{a \in f^{-1}(G)} O_a.$$

Then G_1 is open since it is a union of open sets, & we claim that $f^{-1}(G) = G_1 \cap D(f)$.

To see this, suppose $a \in f^{-1}(G)$. Then since $f^{-1}(G) \subseteq D(f)$, we have $a \in D(f)$.

But we also have $a \in O_a \subseteq G_1$, so $a \in D(f) \cap G_1$.

Thus $f^{-1}(G) \subseteq G_1 \cap D(f)$.

Now suppose that $a \in G_1 \cap D(f)$. Then

since $a \in O_a \subseteq U_a$ we have $f(a) \in f(U_a) \subseteq G$.

Therefore $a \in f^{-1}(G)$, so $G_1 \cap D(f) \subseteq f^{-1}(G)$.

(b) $\Rightarrow$ (a) Suppose that (b) is true. Let $a \in D(f)$.

We must show:

$$\forall \text{ nbhd } V \text{ of } f(a), \ \exists \text{ nbhd } U \text{ of } a \text{ s.t. } f(U) \subseteq V.$$

So, let V be any neighborhood of $f(a)$. By def. of neighborhood, $\exists$ open set G such that $f(a) \in G \subseteq V$. By (b), $\exists$ open set G_1 such that

$$f^{-1}(G) = G_1 \cap D(f). \qquad (*)$$

Now, $a \in D(f)$ & $f(a) \in G$, so $a \in f^{-1}(G)$.

Hence $a \in G_1$. Since G_1 is open, it is therefore a neighborhood of a. Now,

$$f(G_1) = \{f(x) : x \in G_1 \cap D(f)\}.$$

But if $x \in G_1 \cap D(f)$, then $x \in f^{-1}(G)$ by $(*)$, so $f(x) \in G$. Hence

$$f(G_1) \subseteq G.$$

Thus $U = G_1$ is the neighborhood we seek.

(a) $\Leftrightarrow$ (c) Exercise (proof in book). $\blacksquare$

<u>Theorem</u>

If $H \subseteq D(f)$ is connected in $\mathbb{R}^p$ & f is continuous on H, then $f(H)$ is connected in $\mathbb{R}^q$.

<u>Proof:</u>

First restrict the domain of f to the set H, so $D(f) = H$ & f is continuous on its domain [technically this gives a new function with a smaller domain, but it's clear what we mean].

Suppose that (A, B) was a disconnection of $f(H)$ in $\mathbb{R}^q$.

This means that:

(a) A, B are open in $\mathbb{R}^q$

(b) $A \cap f(H) \neq \emptyset$ & $B \cap f(H) \neq \emptyset$

(c) $\big(A \cap f(H)\big) \cap \big(B \cap f(H)\big) = \emptyset$

(d) $\big(A \cap f(H)\big) \cup \big(B \cap f(H)\big) = f(H)$.

By the Global Continuity Theorem, $\exists$ open sets $A_1, B_1 \subseteq \mathbb{R}^p$ such that

$$f^{-1}(A) = A_1 \cap D(f) = A_1 \cap H \quad \& \quad f^{-1}(B) = B_1 \cap D(f) = B_1 \cap H$$

We claim that (A_1, B_1) is a disconnection of H.

Must show:

(a') A_1, B_1 open in $\mathbb{R}^p$ (true by def.)

(b') $A_1 \cap H \neq \emptyset$ & $B_1 \cap H \neq \emptyset$

(c') $(A_1 \cap H) \cap (B_1 \cap H) = \emptyset$

(d') $(A_1 \cap H) \cup (B_1 \cap H) = H$.

Exercise: Prove (b'), (c'), (d')

Example: Show $A_1 \cap H \neq \emptyset$: Since $A \cap f(H) \neq \emptyset$, we know $\exists\ y \in A \cap f(H)$.

Then $y = f(x)$ for some $x \in H$.

Further, $x \in f^{-1}(A)$ because $y = f(x) \in A$.

Hence $x \in f^{-1}(A) = A_1 \cap H$ so $A_1 \cap H \neq \emptyset$.

This proves the first half of (b').

Once (a′) – (d′) have been proved, we conclude that (A_1, B_1) is a disconnection of H.

But H is connected, so this is a contradiction.

Therefore there can't be any disconnections of $f(H)$. $\blacksquare$

Intermediate Value Theorem

Let f map vectors in $\mathbb{R}^p$ to numbers in $\mathbb{R}$.

If f is continuous on a CONNECTED set $H \subseteq D(f)$, then for each real number k such that

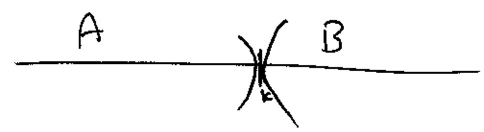
$$\inf\{f(x) : x \in H\} < k < \sup\{f(x) : x \in H\},$$

there exists $x_0 \in H$ such that

$$f(x_0) = k.$$

Proof:

Suppose $k \notin f(H)$. Define



$$A = (-\infty, k) \quad \& \quad B = (k, \infty).$$

Then A, B are open intervals in $\mathbb{R}$ and (A, B) is a disconnection of $f(H)$ (why?).

But H is connected & f is continuous, so $f(H)$ is connected. Therefore this is a contradiction. Hence $k \in f(H)$, which implies $k = f(x_0)$ for some $x_0 \in H$. ∎

Theorem

If $K \subseteq D(f)$ in compact in $\mathbb{R}^p$ & f is continuous on K, then $f(K)$ is compact in $\mathbb{R}^q$.

Proof:

Suppose K is compact. Then K is closed & bounded. We'll show that $f(K)$ is closed & bounded in $\mathbb{R}^q$.

Suppose $f(K)$ was not bounded in $\mathbb{R}^q$. Then for each ball $B_n(0)$ of radius n centered at 0, $\exists$ point $y_n \in f(K) \setminus B_n(0)$. That means $y_n = f(x_n)$ for some $x_n \in K$, and $\|f(x_n)\| \geq n$. Now (x_n) is a sequence in K & K is closed & bounded, so by the Bolzano-Weierstrauss Theorem $\exists$ subsequence (x_{n_k}) that converges, say $x_{n_k} \longrightarrow x \in K$. Since f is continuous, this implies $f(x_{n_k}) \longrightarrow f(x)$. But convergent sequences are bounded, contradicting the fact that $(f(x_{n_k}))$ is an unbounded sequence. Hence $f(K)$ must be bounded.

Now we'll show that $f(K)$ is closed by showing that it contains all its cluster points. Suppose y is a cluster point of $f(K)$. Note we want to prove $y \in f(K)$. Then

$$\forall n \in \mathbb{N},\ \exists z_n \in K \text{ such that } \|y - f(z_n)\| < \frac{1}{n}.$$

Now (z_n) is a sequence in the closed & bounded set K, so it has a convergent subsequence, say $z_{n_k} \longrightarrow z \in K$. Then since f is continuous, we have that $f(z_{n_k}) \longrightarrow f(z)$. But we know that $\|y - f(z_{n_k})\| < \frac{1}{n_k} \longrightarrow 0$, so $f(z_{n_k}) \longrightarrow y$. Since limits are unique, we conclude that $y = f(z)$. Therefore $y \in f(K)$. Thus $f(K)$ is closed. ∎

Max & Min Value Theorem

A continuous real-valued function on a compact set attains its maximum & minimum values.

That is, if $K \subseteq D(f)$ is compact & f is continuous and real-valued, then $\exists\, x_*, x^* \in K$ such that

$$f(x_*) = \inf\{f(x) : x \in K\} \quad \& \quad f(x^*) = \sup\{f(x) : x \in K\}$$

Proof:
Since K is compact in $\mathbb{R}^p$ & f is continuous, $f(K)$ is compact in $\mathbb{R}$. In particular, $f(K)$ is bounded, so $m = \inf\{f(x) : x \in K\}$ & $M = \sup\{f(x) : x \in K\}$ are finite real numbers. By def. of sup,

$$\forall n \in \mathbb{N}\ \exists x_n \in K \text{ s.t. } M - \tfrac{1}{n} < f(x_n) \le M.$$

Therefore $\lim_{n\to\infty} f(x_n) = M$.
Since (x_n) is a sequence in the closed & bounded set K, $\exists$ convergent subsequence, say $x_{n_k} \longrightarrow x^* \in K$. Then since f is continuous, $f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = M$.

A similar argument using inf's shows that x_* exists. ∎