Section 22

Let $f$ be a function from $\mathbb{R}^p$ to $\mathbb{R}^e$ with domain $D(f) \subseteq \mathbb{R}^p$ and range $R(f) \subseteq \mathbb{R}^e$.

Global Continuity Theorem

The following statements are equivalent.

(a) $f$ is continuous at every point $a \in D(f)$
   [we write "$f$ is continuous on $D(f)$" in this case].

(b) If $G \subseteq \mathbb{R}^e$ is open, then
    
    \[ f^{-1}(G) = G_1 \cap D(f) \]
    
    for some open $G_1 \subseteq \mathbb{R}^p$.

(c) If $H \subseteq \mathbb{R}^e$ is closed, then
    
    \[ f^{-1}(H) = H_1 \cap D(f) \]
    
    for some closed $H_1 \subseteq \mathbb{R}^p$. 
Remark: If $\mathbb{D}(f) = \mathbb{R}^p$ then we can rewrite (b) & (c) as

(b') If $G \subseteq \mathbb{R}^e$ is open, then $f^{-1}(G)$ is open in $\mathbb{R}^p$.

(c') If $H \subseteq \mathbb{R}^e$ is closed, then $f^{-1}(H)$ is closed in $\mathbb{R}^p$.

But, if $\mathbb{D}(f) \neq \mathbb{R}^p$ then $f^{-1}(G)$ need not be open. In that case, it only has to be true that $f^{-1}(G) = G \cap \mathbb{D}(f)$, where $G$ is some open set.
Example \( p = q = 1 \) \( \quad \mathcal{D}(f) = \mathbb{R} \).

Note \( \mathcal{D}(f) = \mathbb{R} \).

If \( G = (-1, 4) \) then \( f^{-1}(G) = (-2, 2) \) is open.

If \( H = [-1, 4] \) then \( f^{-1}(H) = [-2, 2] \) is closed.

Ex: Same except restrict to \( \mathcal{D}(f) = [0, \infty) \)

Then \( f^{-1}(G) = [0, 2) \) is not open

\( = (-2, 2) \cap \mathcal{D}(f) \)

\( G \) open
Proof of Theorem

(a) ⇒ (b) Suppose \( f \) is continuous on \( D(f) \).

Let \( G \subseteq \mathbb{R}^n \) be any open set. Recall that

\[
    f^{-1}(G) = \{ x \in D(f) : f(x) \in G \}.
\]

Let \( a \in f^{-1}(G) \). Then \( f(a) \in G \), & \( G \) is open so it is a neighborhood of \( f(a) \). By def. of continuity, \( \exists \) neighborhood \( U_a \) of \( a \) such that \( f(U_a) \subseteq G \). By def. of neighborhood, \( \exists \) open set \( O_a \) such that

\[
    a \in O_a \subseteq U_a.
\]

Define

\[
    G_1 = \bigcup_{a \in f^{-1}(G)} O_a.
\]

Then \( G_1 \) is open since it is a union of open sets, \& we claim that \( f^{-1}(G) = G_1 \cap D(f) \).

To see this, suppose \( a \in f^{-1}(G) \). Then
since $f^{-1}(G) \subseteq D(f)$, we have $a \in D(f)$.

But we also have $a \in O_a \subseteq G$, so $a \in D(f) \cap G$.

Thus $f^{-1}(G) \subseteq G \cap D(f)$.

Now suppose that $a \in G \cap D(f)$. Then since $a \in O_a \subseteq U_a$, we have $f(a) \in f(U_a) \subseteq G$.

Therefore $a \in f^{-1}(G)$, so $G \cap D(f) \subseteq f^{-1}(G)$.
(b) \implies (a) \quad \text{Suppose that (b) is true. Let } a \in D(f).

We must show:

\text{An nhd } V \text{ of } f(a), \quad \exists \text{nhd } U \text{ of } a \text{ s.t. } f(U) \subseteq V.

So, let \( V \) be any nhd of \( f(a) \). By def. of nhd, \( \exists \) open set \( G \) such that \( f(a) \in G \subseteq V \).

By (b), \( \exists \) open set \( G_i \) such that
\[
f^{-1}(G) = G_i \cap D(f).
\]

(\star)

Now, \( a \in D(f) \) \& \( f(a) \in G \), so \( a \in f^{-1}(G) \).

Hence \( a \in G_i \). Since \( G_i \) is open, it is therefore a nhd of \( a \). Now,
\[
f(G_i) = \{ f(x) : x \in G_i, x \in D(f) \}.
\]

But if \( x \in G_i \cap D(f) \), then \( x \in f^{-1}(G) \) by (\star), so \( f(x) \in G \). Hence
\[
f(G_i) \subseteq G.
\]

Thus \( U = G_i \) is the nhd we seek.
(a) \implies (c): Exercise (proof in book).

\textbf{Theorem}

If \( H \subseteq \text{Dom}(f) \) is connected and \( f \) is continuous on \( H \),
then \( f(H) \) is connected in \( \mathbb{R}^n \).

\textbf{Proof:}

First restrict the domain of \( f \) to \( H \) and set \( H \),
so \( \text{Dom}(f) = H \) and \( f \) is continuous on its domain.

[Technically this gives a new function with a smaller domain,
but it's clear what we mean.]

Suppose that \((A, B)\) was a disconnection of \( f(H) \) in \( \mathbb{R}^n \).
This means that:

(a) \( A, B \) are open in \( \mathbb{R}^n \)

(b) \( A \cap f(H) \neq \emptyset \) and \( B \cap f(H) \neq \emptyset \)

(c) \( (A \cap f(H)) \cap (B \cap f(H)) = \emptyset \)

(d) \( (A \cap f(H)) \cup (B \cap f(H)) = f(H) \).
By the Global Continuity Theorem, \( f \) open sets \( A, B \subset \mathbb{R}^n \) such that

\[
 f^{-1}(A) = A \cap D(f) \quad \& \quad f^{-1}(B) = B \cap D(f),
\]

We claim that \((A, B)\) is a disconnection of \( H \).

Must show:

\( a' \) \( A, B \), open in \( \mathbb{R}^n \) (true by def.)

\( b' \) \( A \cap H \neq \emptyset \) \( \& \) \( B \cap H \neq \emptyset \)

\( c' \) \( (A \cap H) \cap (B \cap H) = \emptyset \)

\( d' \) \( (A \cap H) \cup (B \cap H) = H \).

**Exercise:** Prove \( b', c', d' \)

**Example:** Since \( A \cap f(H) \neq \emptyset \), we know \( \exists y \in A \cap f(H) \).

Then \( y = f(x) \) for some \( x \in H \).

Further, \( x \in f^{-1}(A) \) because \( y = f(x) \in A \).

Hence \( x \in f^{-1}(A) = A \cap H \) so \( A \cap H \neq \emptyset \).

This proves the first half of \( b' \).
Once $(a') - (d')$ have been proved, we conclude that $(A, B)$ is a disconnection of $H$.

But $H$ is connected, so $(A, B)$ is a contradiction. Therefore there can't be any disconnections of $f(t)$. \[\Box\]
Intermediate Value Theorem

Let $f$ map vectors in $\mathbb{R}^p$ to numbers in $\mathbb{R}$. If $f$ is continuous on a set $H \subseteq \text{DOM}(f)$, and for each real number $k$ such that

$$\inf \{f(x) : x \in H\} < k < \sup \{f(x) : x \in H\},$$

there exists $x_0 \in H$ such that

$$f(x_0) = k.$$

Proof: Suppose $k \notin f(H)$. Define $A = (-\infty, k]$ and $B = (k, \infty)$. Then $A, B$ are open intervals in $\mathbb{R}$ and $(A, B)$ is a disconnection of $f(H)$ (why?). But $H$ is connected and $f$ is continuous, so $f(H)$ is connected. Therefore $\exists$ is a contraction. Hence $k \in f(H)$, which implies $k = f(x_0)$ for some $x_0 \in H$. \[\square\]
Theorem
If \( K \subseteq \mathbb{D}(f) \) is compact in \( \mathbb{R}^p \) and \( f \) is continuous on \( K \), then \( f(K) \) is compact in \( \mathbb{R}^q \).

Proof:
Suppose \( K \) is compact. Then \( K \) is closed and bounded.
We will show that \( f(K) \) is closed and bounded in \( \mathbb{R}^q \).

Suppose \( f(K) \) was not bounded in \( \mathbb{R}^q \). Then for each ball \( B_n(o) \) of radius \( n \) centered at \( o \),
\( \exists \) point \( y_n \in f(K) \setminus \overline{B_n(o)} \). That means
\( y_n = f(x_n) \) for some \( x_n \in K \), and \( \| f(x_n) \| \geq n \).
Now \( (x_n) \) is a sequence in \( K \) and \( K \) is closed and bounded,
so by the Bolzano-Weierstrass Theorem \( \exists \) subsequence \( (x_{n_k}) \) that converges, say \( x_{n_k} \rightarrow x \in K \).
Since \( f \) is continuous, \( f(x_{n_k}) \rightarrow f(x) \).
But convergent sequences are bounded, contradicting
A fact that \((f(x_n))\) is an unbounded sequence. Hence \(f(K)\) must be bounded.

Now we'll show that \(f(K)\) is closed by showing that it contains all its cluster points. Suppose \(y\) is a cluster point of \(f(K)\). Then

\[ \forall n \in \mathbb{N}, \exists z_n \in K \text{ such that } \|y - f(z_n)\| < \frac{1}{n}. \]

Now \((z_n)\) is a sequence in a closed & bounded set \(K\), so it has a convergent subsequence, say \(z_{n_k} \rightarrow z \in K\). Then since \(f\) is continuous, we have that \(f(z_{n_k}) \rightarrow f(z)\). But we know that \(\|y - f(z_{n_k})\| < \frac{1}{n_k} \rightarrow 0\), so \(f(z_{n_k}) \rightarrow y\).

Since limits are unique, we conclude that \(y = f(z)\). Therefore \(y \in f(K)\). Thus \(f(K)\) is closed.
Max & Min Value Theorem

A continuous real-valued function on a compact set attains its maximum & minimum values.

That is, if \( K \subseteq \mathbb{D}(f) \) is compact & \( f \) is continuous and real-valued, then \( \exists x_*, x^* \in K \) such that \( f(x_*) = \inf \{ f(x) : x \in K \} \) & \( f(x^*) = \sup \{ f(x) : x \in K \} \).

Proof:
Since \( K \) is compact in \( \mathbb{R}^n \) & \( f \) is continuous, \( f(K) \) is compact in \( \mathbb{R} \). In particular, \( f(K) \) is bounded, so \( m = \inf \{ f(x) : x \in K \} \) & \( M = \sup \{ f(x) : x \in K \} \) are finite real numbers. By def. of \( \sup \),

\[
\forall n \in \mathbb{N} \ \exists x_n \in K \ \text{s.t.} \ M - \frac{1}{n} < f(x_n) \leq M.
\]

Therefore \( \lim_{n \to \infty} f(x_n) = M \).
Since \( (x_n) \) is a sequence in the closed & bounded set \( K \), \( \exists \) convergent subsequence, say \( x_{n_k} \to x^* \in K \). Then since \( f \) is continuous, \( f(x^*) = \lim_{k \to \infty} f(x_{n_k}) = M \).

A similar argument using \( \inf \)’s shows that \( x_* \) exists. \( \square \)