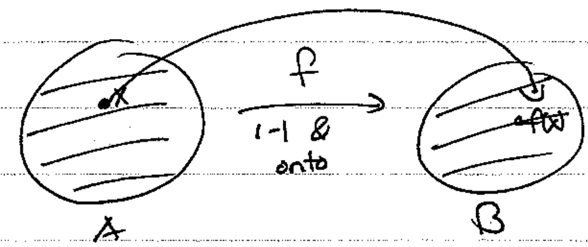
# 3. Finite & Infinite Sets




Every element of A lined up 1-1 with elements of B.

**Definition**

If $\underbrace{f: A \to B}$ is a bijection, then we also say that f is a 1-1 correspondence, and that the sets A & B can be put into 1-1 correspondence with each other.

$D(f) = A$
$R(f) = B$
$f$ is injective

Intuition: If $\exists$ bijection $f: A \to B$ then A & B have the "same size" or the "same # of elements". We say they have the same cardinality

**Definition**

$S_n = \{1, 2, \dots, n\}$ is an initial segment of $\mathbb{N}$.

(a) If $\exists$ bijection $f: A \to S_n$ (or $f: S_n \to A$)

then A has n elements & is a finite set.

(b) If $\exists$ bijection $f: A \to \mathbb{N}$ (or $f: \mathbb{N} \to A$)

then A is denumerable.

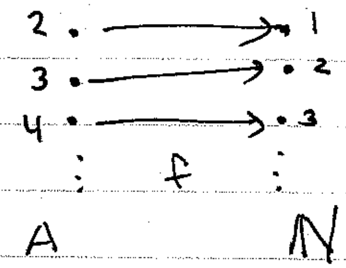
(c) A is countable if it is finite or denumerable

(d) A is uncountable if it is not countable.

<u>Homework</u>: $m \neq n \Rightarrow \nexists$ bijection $f: S_m \to S_n$.

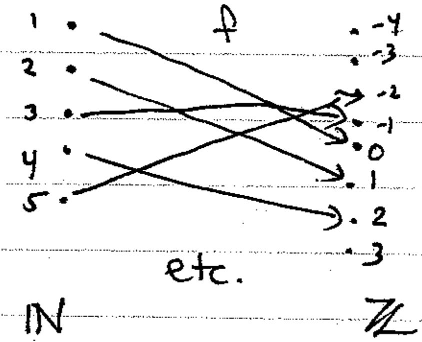
<u>Examples</u>

(a) $A = \{2, 3, 4, \ldots\}$ is denumerable




$f(n) = n-1$ is a bijection of $A$ onto $\mathbb{N}$.

(b) $\mathbb{Z}$ is denumerable




$f: \mathbb{N} \to \mathbb{Z}$ is a bijection.

Could give a formula ~~for~~ f if desired

$$f(n) = \begin{cases} n/2, & n \text{ even} \\ -(n-1)/2, & n \text{ odd} \end{cases}$$

(c) $\mathbb{Q}$ is denumerable

$$\begin{array}{ccccc} & 1 & 2 & 6 & 7 \\ & 1/1 & 1/2 & 1/3 & 1/4 \\ 3 & 2/1 & \cancel{2/2} & 2/3 \; 8 & \cancel{2/4} \\ 4 & 3/1 & 3/2 \; 9 & \cancel{3/3} & 3/4 \\ 10 & 4/1 & \cancel{4/2} & & \\ 11 & 5/1 & & & \end{array}$$

$f(1) = 1/1$

$f(2) = 1/2$

$f(3) = 2/1$

$\vdots$

$f: \mathbb{N} \to \mathbb{Q}^+$

$f$ is a bijection

$\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$

(d) Any countable union of countable sets is countable

Proof:

Suppose $S = \bigcup_{n=1}^{\infty} A_n$ with each $A_n$ countable.

Let $A_n = \{a_{n1}, a_{n2}, \dots\}$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \\ \vdots & & & \end{matrix}$$

Cross off duplicates & define the bijection similarly as for $\mathbb{Q}$

<u>Theorem</u> $\mathbb{R}$ is uncountable. (but $\mathbb{Q}$ is "dense" in $\mathbb{R}$!)

Decimal expansions, nonuniqueness (just omit 0's & 9's)

<u>Homework</u>: $m \neq n \Rightarrow \nexists$ bijection $f: S_m \to S_n$.

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~~Axiom: <u>Well-ordering property of $\mathbb{N}$</u>~~

~~Every subset of $\mathbb{N}$ contains a least element.~~

~~Axiom: <u>Axiom of Choice</u> (Weak Form)~~

~~Every infinite set contains a denumerable subset.~~

Examples A set can be in 1-1 correspondence with a proper subset of itself.

(a) $f: [0,2] \longrightarrow [0,1]$ is a bijection
$x \longmapsto x/2$

(b) $f: (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ is a bijection
$x \longmapsto \tan x$

(c) Exercise: Find a bijection $f: [0,1] \to (0,1)$ Hint: Can't be a continuous function

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$\mathbb{N}$ & $\mathbb{R}$ are both infinite, yet they have different "sizes" (cardinality).

Completely nonintutive since rationals (countable) & irrationals (uncountable) are both dense in $\mathbb{R}$!

Moreover, $\exists\ \infty$ many different cardinalities!

Example: Let $F = \{\text{all functions } f: \mathbb{R} \to \mathbb{R}\}$.
Then $\nexists$ bijection $\alpha: \mathbb{R} \longrightarrow F$.

Proof:
Suppose there was. $x \longleftrightarrow f_x$.
Define $g(x) = f_x(x+1)$. Then $g \neq f_x$ for any $x \in \mathbb{R}$, contradiction. $\boxtimes$

Repeat: $\mathcal{F} = \{\text{functions } \alpha : F \to F\}$

has cardinality $> F$ !

Possible cardinalities: $1, 2, \ldots$ (finite sets)

$|\mathbb{N}|, |\mathbb{R}|, |F|, |\mathcal{F}|, \ldots$

Q. Are there only countably many cardinalities?

## Continuum Hypothesis

There are no cardinalities between $|\mathbb{N}|$ & $|\mathbb{R}|$.

This is a set theory axiom: cannot be proved or disproved from the other axioms.

Either assume it is true & prove theorems, or assume it is false & prove theorems.
(Doesn't matter for real analysis).

Other axioms we'll need:

<u>Axiom</u>: <u>Well-ordering property of $\mathbb{N}$</u> (not a set theory axiom)

Every subset of $\mathbb{N}$ contains a least element.

<u>Axiom</u>: <u>Axiom of Choice (weak form)</u> (stronger form is a set theory axiom)

Every infinite set contains a denumerable subset.

<u>Exercise</u>: Find a denumerable subset of: $\mathbb{R}$
$\mathcal{F}$
$\mathcal{J}$

<u>Q</u>. Is $|\mathbb{R}| = |\mathbb{C}|$ ?

<u>Yes</u>. Space-filling curves.