7. Cuts, Intervals, & The Cantor Set

**Definition**

Let $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$. Then $(A, B)$ is a cut of:

$A \cap B = \emptyset$, $A \cup B = \mathbb{R}$, $a \in A$ & $b \in B \Rightarrow a < b$.

**Example:**

$A = (-\infty, x) = \{y : y < x\}$

$B = [x, \infty) = \{z : z \geq x\}$

**Claim:** Each cut is determined by a unique real number.

**Cut Property**

If $(A, B)$ is a cut in $\mathbb{R}$ then there is a unique $\xi \in \mathbb{R}$ s.t.

$a \leq \xi \leq b \quad \forall a \in A \quad \& \quad b \in B$

**Proof:**

A is bounded above (by any element of $B$)

$B$ is below $A$

Set $\xi = \sup A$. Then $a \leq \xi \leq b$ for every $b \in B$

Since every element of $B$ is an upper bound for $A$ & $\xi$ is the least upper bound for $A$, we must have $\xi = b$

**Purpose of Cuts**

Used to construct the real nos. 1 from set theory axioms — we'll take the construction of $\mathbb{R}$ as given.
Suppose there also existed \( y \) st. \( y \geq a \ \forall a \in A \) \\
\( y \leq b \ \forall b \in B \).
Then \( y \) is an upper bound for \( A \) so \( \bar{z} \leq y \).

Suppose \( \bar{z} < y \). Then \( \exists x \) st. \( \bar{z} < x < y \).

Then \( x > a \ \forall a \in A \Rightarrow x \notin A \) \\
\( x < b \ \forall b \in B \Rightarrow x \notin B \) \ \ \ \text{Contradiction} \\
\text{A} \cup \text{B} = \text{R}.

Here \( \bar{z} = y \) is the only possibility. \( \Box \)
Cells, Rays, Intervals: Particular subsets of $\mathbb{R}$

Rays: $(a, \infty)$ $[a, \infty)$ $(-\infty, a)$ $(-\infty, a]$
open closed open closed

Cells: $(a, b)$ $[a, b]$ $(a, b]$ $[a, b]$
open closed half-open open

Interval = cell, ray, or $\mathbb{R}$

Unit cell $[0, 1]$

Intervals $I_1, I_2, \ldots$ are nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$

Ex: $I_n = (0, \frac{1}{n})$ is nested. Note $\bigcap I_n = \emptyset$
$J_n = [0, \frac{1}{n}]$ " " $\bigcap J_n = \{0\}$
$K_n = [-\frac{1}{n}, 1+\frac{1}{n}]$ " " $\bigcap K_n = [0, 1]$.

Nested Cells Property
Let $I_n = [a_n, b_n]$ be such that the seq. $I_1, I_2, \ldots$ is nested. Then $\bigcap I_n \neq \emptyset$. 
Proof:
Since \([a_n, b_n] = I_n \subseteq I_1 = [a_1, b_1]\) we must have
\(a_n \geq a, \forall n\). Let \(\bar{z} = \sup \{a_n : n \in \mathbb{N}\}\).

Claim: \(\bar{z} \leq b_1 \forall n\).

Suppose not. Then \(\exists m \text{ st. } b_m < \bar{z}\).

But \(\bar{z} = \sup \{a_n\}\) so \(\exists n \text{ st. } b_m < a_n \leq \bar{z}\).

If \(m \geq n\): \(b_m < a_n \leq a_m\) Contradiction

If \(m < n\): \(b_n \leq b_m < a_n\) Contradiction

so \(\bar{z} \leq b_n \forall n\). Hence \(\bar{z} \in [a_n, b_n] \forall n\),
so \(\bar{z} \in \cap [a_n, b_n]\).

Note: Nested Cells Property can fail if \(\cap\) sets are not closed.

\[\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset\]
Cantor Set

Define sets $F_0, F_1, F_2, \ldots$ by the following procedure.

$F_0 = [0, 1]$

Now create $F_1$ by removing the "middle third" from $F_0$:

$F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

Note $F_1$ consists of 2 intervals of length $\frac{1}{3}$ each.

Now create $F_2$ by removing the middle third from each of the intervals that comprise $F_1$:

$F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

Note $F_2$ consists of 4 = $2^2$ intervals each of length $\frac{1}{9} = \frac{1}{3^2}$.

Keep repeating this process:

$F_3$

$F_4$ consists of 8 = $2^3$ intervals each of length $\frac{1}{27} = \frac{1}{3^3}$. 
Keep going ... $F_n$ looks like

$$F_n = \bigcup_{\text{particular values of } k} \left[ \frac{k}{3^n}, \frac{k+1}{3^n} \right]$$

i.e., $F_n$ is a union of $2^n$ intervals, each of length $\frac{1}{3^n}$, and if we wanted, we could specifically identify the $k$'s in the union above.

Note all sets $F_n$ are nested, i.e.,

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$$

Also,

- total length of intervals in $F_0 = 1 \times 1 = 1$
- $F_1 = 2 \cdot \frac{1}{3} = \frac{2}{3}$
- $F_2 = 4 \cdot \frac{1}{9} = \frac{4}{9} = \left(\frac{2}{3}\right)^2$
- $\vdots$
- $F_n = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$
What if we do this "forever" - is there anything left?

What is left?

Stuff in $F_0$ but not $F_1$ is removed at the first step - only things in both $F_0$ & $F_1$ remain.

Stuff in $F_1$ but not $F_2$ is removed at the 2nd step - only things in all of $F_0$, $F_1$, $F_2$ remain.

: 

Only stuff in $F_1$ remains at the 3rd step.

: 

Stuff that remains after every step is

$$ F = \bigcap_{n=1}^{\infty} F_n $$

This is the Cantor set.

Q. Is there anything in the Cantor set??

We have

$$ F \subseteq F_n \quad \text{for every } n, $$

but perhaps $F$ is just an empty set??
Each $F_n$ consists of $2^n$ intervals. There are $2 \cdot 2^n = 2^{n+1}$ endpoints of these intervals.

For each $n$, the $2^{n+1}$ endpoints of the intervals that comprise $F_n$ belong to $F_1$.

So, $F$ does contain infinitely many points.

There are countably many endpoints of $2$ intervals.
in $\mathbb{R}^n$, so $F$ has at least countably many points in it.

Q. Are these all the elements of $F$?

To answer this, we will use ternary expansions.

Note:
$F$ itself does NOT have any "endpoints." $\frac{1}{3}$ is not an "endpoint" of $F$. $\frac{1}{3}$ is an endpoint of an interval in $F$, but $F$ itself contains no intervals.

Ex: Prove that $\#$ interval $(a,b)$ such that $(a,b) \subseteq F$. 
Decimal Expansions

Recall that when we wrote

\[ x = 0.d_1d_2d_3 \ldots \]  

(each digit is one of 0, 1, \ldots, 9)

we mean

\[ x = \sum_{k=1}^{\infty} \frac{d_k}{10^k} \]  

Exercise: This series converges for every choice of digits \( d_k \).

Why?

Example

Let's prove that \( \frac{1}{3} = .333\ldots \). That is, let

\[ x = \sum_{k=1}^{\infty} \frac{3}{10^k} \]

and let's prove that \( x = \frac{1}{3} \). We have

\[ 10x = 10 \sum_{k=1}^{\infty} \frac{3}{10^k} \]

\[ = \sum_{k=1}^{\infty} \frac{3}{10^{k-1}} = \frac{3}{10^0} + \sum_{k=1}^{\infty} \frac{3}{10^k} = 3 + \frac{3}{10} + \frac{3}{100} + \ldots \]

\[ = 3 + x \]

So, whatever \( x \) is, it satisfies the equation

\[ 10x = 3 + x \]
$10x = 3 + x$

Hence $9x = 3$ so $x = \frac{1}{3}$.

**Exercise**

Some numbers have two decimal expansions. Show that

$1.000\ldots = 0.999\ldots$

That is, show that

$$1 = \sum_{k=1}^{\infty} \frac{9}{10^k}$$

(use the same technique as in the example).

**Remark:**

The numbers that have two expansions are the ones that have an expansion ending in $\infty$ many zeros. E.g.,

$0.7346 = 0.7345999\ldots$
What does its decimal expansion tell you?

Consider \(x = .314159\ldots\)

3 tells you you're in (0, 1) interval

\[\begin{array}{c}
0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \\
\hline
\end{array}\]

1 tells you you're in (0.3, 0.4) interval

\[\begin{array}{c}
.30 & .31 & .32 & .33 & .34 & .35 & .36 & .37 & .38 & .39 & 1 \\
\hline
\end{array}\]

4 tells you you're in (0.310, 0.315) interval

\[\begin{array}{c}
.310 & .311 & .312 & .313 & .314 & .315 & .316 & .317 & .318 & .319 & .32 \\
\hline
\end{array}\]

etc. At each stage you divide the interval into 10 parts and identify which part you are in. For instance, for "endpoint" numbers, like .3, .32, .314, \ldots there are two choices of subinterval to go in, so this gives \(\aleph_0\) two decimal expansions.

Isn't this reminiscent of a Cantor set construction, except using tens instead of twos?
Ternary Expansions

There's nothing special about the number 10, we can base "decimal expansions" on any positive integer. In particular, ternary expansions use base 3.

We write
\[ x = 0.d_1d_2d_3 \ldots \text{ base 3} \quad \text{(each } d_k \text{ is one of } 0, 1, 2) \]

to mean
\[ x = \sum_{k=1}^{\infty} \frac{d_k}{3^k}. \]

Exercise: \( \frac{1}{3} = .1000 \ldots \text{ base 3} = .0222 \ldots \text{ base 3} \).

Some numbers have two ternary expansions; in this case one expansion ends with infinitely many zeros, and the other with infinitely many 2's.
Exercise \[ \frac{1}{4} = 0.020202 \ldots \text{ base 3} \]

What points are in \( \mathcal{C} \) Cantor set?

Write \( X = 0.d_1d_2d_3 \ldots \text{ base 3} \).

Stage 1:

\[ \begin{array}{c}
\text{d}_1 = 0 \\
0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1 \\
\text{d}_1 = 2
\end{array} \]

These points are removed. They all have \( d_1 = 1 \).

The endpoints, \( \frac{1}{3} \) & \( \frac{2}{3} \) have two ternary expansions:

\[ \frac{1}{3} = 0.1000 \ldots = 0.0222 \ldots \]

\[ \frac{2}{3} = 0.2000 \ldots = 0.1222 \ldots \]

They are not removed.

So: As long as \( x \) has a ternary expansion \( 0.12 \ldots \) has either \( d_1 = 0 \) or \( d_1 = 2 \), then \( x \) is not removed in Stage 1.
Stage 2:

\[ d_1 = 0 \quad d_1 = 2 \]
\[ \{ \quad \{ \]
\[ d_2 = 1 \quad d_2 = 1 \]

In stage 2, all remaining points that have \( d_2 = 1 \) (except for the endpoints). Precisely, as long as there's a ternary expansion that has either \( d_2 = 0 \) or \( d_2 = 2 \), the point is not removed.

Now repeat. If

\[ x = 0.d_1d_2d_3\ldots \text{ base } 3 \]

\& \( d_1, \ldots, d_n = 0 \) or \( 2 \) (for at least one of \( x \)'s ternary expansions) then \( x \in F \) is not removed in the first \( n \) stages, i.e., \( x \in F_n \).

Conclude: if there's a ternary expansion such that every \( d_k \) is 0 or 2, then \( x \) is in
every $F_n$ and hence is in $F = \cap F_n$.

Example: $\frac{1}{4} = 0.020202 \ldots$ base 3

so $\frac{1}{4} \in F$, even though $\frac{1}{4}$ is not an endpoint of any $F_n$!

How many points are in $F$?

Each $x \in F$ can be written

$$X = 0.d_1d_2d_3 \ldots \text{ base 3}$$

where each $d_k$ is 0 or 2. Define

$$d'_k = \begin{cases} 1 & \text{if } d_k = 2 \\ 0 & \text{if } d_k = 0 \end{cases}$$

Define

$$f(x) = 0.d'_1d'_2d'_3 \ldots \text{ base 2} = \sum_{k=1}^{\infty} \frac{d'_k}{2^k}.$$
Example

\[ f(\frac{1}{4}) = f(0.020202\ldots \text{ base } 3) \]

\[ = 0.010101\ldots \text{ base } 2 \]

\[ = \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \ldots \]

\[ = \sum_{k=1}^{\infty} \frac{1}{4^k} \]

\[ = \frac{1}{3}. \]

Claim: \( f \) maps \( \mathbb{F} \) onto \([0, 1]\).

Proof:
Suppose \( y \in [0, 1] \). Then \( y \) has a base 2 expansion

\[ y = 0.e_1e_2e_3\ldots \text{ base } 2 \quad e_k = 0 \text{ or } 1. \]

Set

\[ d_k = \begin{cases} 2 & \text{if } e_k = 1 \\ 0 & \text{if } e_k = 0 \end{cases} \]

and

\[ x = .d_1d_2d_3\ldots \text{ base } 3. \]

Then \( x \) has a ternary expansion that contains
only 0's & 2's, so \( X \in F \). Further, by definition of \( f \) we have \( f(x) = y \). Hence \( f \) maps onto all of \([0, 1]\).

Exercise

Even without knowing whether \( f \) is injective, show that \( 2^n \) implies that \( F \) is uncountable.

More generally, show that if \( X, Y \) are sets & \( Y \) is uncountable, if \( f \) surjection \( f : X \to Y \) then \( X \) must be uncountable.

THE CANTOR SET IS UNCOUNTABLE!

Compare: The endpoints of the \( F_n \) sets are in \( F \), but there are only countably many of these. Hence, “most” points in \( F \) are not endpoints of any \( F_n \).
How “long” is the Cantor Set?

Note: Each \( F_n \) is a union of finitely many intervals, but \( F \), whatever it is, is not a union of intervals. Hence we can’t really talk about \( F \)’s “total length.” Still, if it did make sense to talk about \( F \)’s “length,” what would it be?

\[
\begin{align*}
F \subseteq F_0 \quad &\Rightarrow \quad \text{length}(F) \leq \text{length}(F_0) = 1 \\
F \subseteq F_1 \quad &\Rightarrow \quad \text{length}(F) \leq \text{length}(F_1) = \frac{2}{3} \\
\vdots \\
F \subseteq F_n \quad &\Rightarrow \quad \text{length}(F) \leq \text{length}(F_n) = \left(\frac{2}{3}\right)^n
\end{align*}
\]

This is true for every \( n \), so since \( \left(\frac{2}{3}\right)^n \to 0 \) as \( n \to \infty \) we can only have

\[
\text{length}(F) = 0!
\]

Q. Is it possible for a nonempty set to have “length 0”? [We still haven’t defined exactly what “length” of an arbitrary]
set means — this is what measure theory is about, taught at GA Tech in a graduate course MATH 6327.

A. Yes, there are lots of nonempty sets that have "length" zero.

Examples: a single point \( \{x\} \)

Finitely many points: \( \{x_1, \ldots, x_n\} \)

Countably many points: \( \{x_1, x_2, \ldots\} \)

Why should a set like \( S = \{x_1, x_2, \ldots\} \) have length zero?

Intuitively, since \( S = \bigcup_{n=1}^{\infty} \{x_n\} \), we should hope that

\[
\text{(x) } \text{length}(S) = \sum_{n=1}^{\infty} \text{length}(\{x_n\}) = \sum_{n=1}^{\infty} 0 = 0
\]

Warning: What seems intuitively clear in measure theory is not always true. In particular, there is no way to extend the definition of length of an interval to length of arbitrary subsets of \( \mathbb{R} \) so that
\[ A \cap B = \emptyset \implies \text{length}(A \cup B) = \text{length}(A) + \text{length}(B) ! \]

That is, there exist disjoint sets \( A, B \) such that the "length" of their union is greater than the sum of their lengths.

Even so, equation (*) is valid and we conclude that

If \( S \subseteq \mathbb{R} \) is countable then \( \text{length}(S) = 0 \).

This is not a surprise — the surprise is that \( \exists \) uncountable sets with length zero!

The Cantor set is an uncountable set that has zero total length.

The Cantor set is a very interesting set & we will see more of its surprising properties later.
Exercise

Since the set \( \mathbb{Q} \) of all rationals is countable, we can make a list of all the rational numbers in \( \mathbb{R} \).

Let
\[
\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}
\]
be one such ordering of \( \mathbb{Q} \). Fix any \( \varepsilon > 0 \), and let \( I_k \) be an interval of length \( \varepsilon / 2^k \) centered at \( q_k \):
\[
I_k = \left[q_k - \frac{\varepsilon}{2^k}, q_k + \frac{\varepsilon}{2^k}\right].
\]

Let \( E = \bigcup_{k=1}^{\infty} I_k \). Show that

a. \( \mathbb{Q} \subseteq E \)

b. \( \text{length}(E) \leq \varepsilon \) (note \( I_k \) might overlap, so only can say \( \leq \), not =).

Try to explain how this is possible - the rationals are "dense" in \( \mathbb{R} \) yet have been covered by a set with total length at most \( \varepsilon \). What does \( \mathbb{R} \setminus E \) look like?
like - does it contain any intervals?

Exercise
Make a Cantor-like set by removing at each stage a set of a different relative length. For example, instead of always removing the middle \(\frac{1}{3}\) of the interval, you could remove the middle \(\frac{1}{n}\) at stage \(n\). Show that by choosing correct relative lengths you can create a Cantor-like set \(C\) that is uncountable, has positive total length, yet contains no intervals.