TOPOLOGY OF CARTESIAN SPACES

8. Vector & Cartesian Spaces

Definition

A vector space is a set V together with two operations:

$$+ : V \times V \longrightarrow V \quad \text{vector addition}$$
$$\cdot : \mathbb{R} \times V \longrightarrow V \quad \text{scalar mult.}$$

such that

(a) $x + y = y + x \quad \forall x, y \in V$
(b) $(x+y) + z = x + (y+z) \quad \forall x, y, z \in V$
(c) $\exists 0 \in V \ \forall x \in V \quad x + 0 = x$
(d) $\forall x \in V \ \exists (-x) \in V$ s.t. $x + (-x) = 0$

(e) $1x = x \quad \forall x \in V$
(f) $a(bx) = (ab)x \quad \forall a, b \in \mathbb{R} \ \forall x \in V$
(g) $(a+b)x = ax + bx, \quad a(x+y) = ax + ay \quad \forall a, b \in \mathbb{R}, \forall x, y \in V.$

[$\mathbb{R}$ can be replaced by any FIELD]

Examples: $\mathbb{R}$

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$$

$\mathbb{R}^S$

If S is any set, then we define $\mathbb{R}^S$ to be the set of all functions that map S into $\mathbb{R}$:

$$\mathbb{R}^S = \{ f : f : S \to \mathbb{R} \}$$

This is a vector space. The rule for vector addition is that if $f, g : S \to \mathbb{R}$ then $f+g : S \to \mathbb{R}$ is given by

$$(f+g)(x) = f(x) + g(x), \quad x \in \mathbb{R}$$

and if $f : S \to \mathbb{R}$ & $c \in \mathbb{R}$ then $cf : S \to \mathbb{R}$ is

$$(cf)(x) = c f(x), \quad x \in \mathbb{R}.$$

We'll give several examples of such spaces.

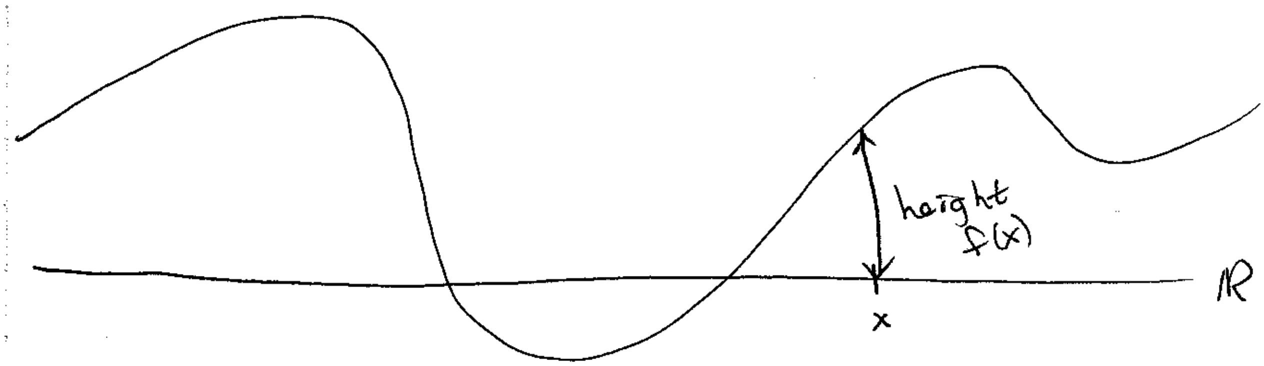
$\mathbb{R}^{\mathbb{R}}$

By definition, $\mathbb{R}^{\mathbb{R}}$ is the set of all functions that map real nos. to real nos.:

$$\mathbb{R}^{\mathbb{R}} = \{ f : f : \mathbb{R} \to \mathbb{R} \}$$

We've seen this set before, we called it F or $\mathcal{F}$.

If $f \in \mathbb{R}^{\mathbb{R}}$ then we can describe it by its graph:



domain in $\mathbb{R}$, range is $\mathbb{R}$

Abusing notation, we have $e^x \in \mathbb{R}^{\mathbb{R}}$, $\sin x \in \mathbb{R}^{\mathbb{R}}$, etc.

<u>Exercise</u>

Show that $\mathbb{R}^{\mathbb{R}}$ is a vector space. The operations are defined as follows: if $f, g \in \mathbb{R}^{\mathbb{R}}$ then $f+g$ is the function whose rule is

$$(f+g)(x) = f(x) + g(x), \quad x \in \mathbb{R}$$

Likewise, if $f \in \mathbb{R}^{\mathbb{R}}$ & $c \in \mathbb{R}$ then cf is the function whose rule is

$$(cf)(x) = c\,f(x), \quad x \in \mathbb{R}.$$

The <u>zero vector</u> in $\mathbb{R}^{\mathbb{R}}$ is the <u>function 0</u> (note that we are abusing notation & writing the symbol 0 to denote a function) whose rule is

$$0(x) = 0, \quad x \in \mathbb{R}$$

zero function

zero number

$\mathbb{R}^{\mathbb{N}}$

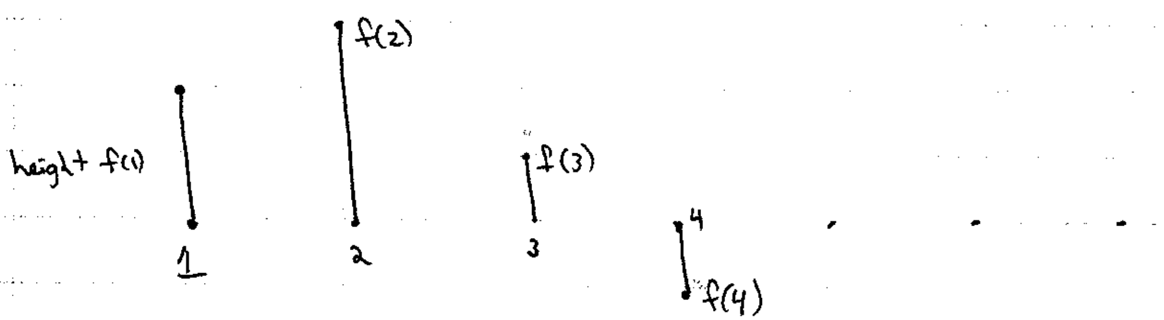
By definition, $\mathbb{R}^{\mathbb{N}}$ is the set of all functions that map $\mathbb{N}$ into $\mathbb{R}$:

$$\mathbb{R}^{\mathbb{N}} = \{ f : f: \mathbb{N} \to \mathbb{R} \}$$

Let f be an element of $\mathbb{R}^{\mathbb{N}}$, i.e., $f: \mathbb{N} \to \mathbb{R}$.

So f maps natural numbers to real numbers.

If we know what $f(1), f(2), f(3), \ldots$ are, then we know f completely. We can think of f as a "discrete" or "digital" function:



domain is $\{1, 2, 3, \ldots\}$ range is $\mathbb{R}$

Space of sequences

Let

$$S = \{ x = (x_1, x_2, x_3, \dots) \ : \ x_1, x_2, \dots \in \mathbb{R} \}$$

That is, S is the set of all infinite sequences of real numbers.

Exercise

S is a vector space under the operations

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$$

$$c(x_1, x_2, \dots) = (cx_1, cx_2, \dots)$$

The zero vector is

$$0 = (0, 0, 0, \dots)$$

Note that we usually abuse notation & use the same symbol 0 to denote two different things: it can be either the zero number or the zero sequence. We tell the difference by looking at the context in which the symbol is used.

$\mathbb{R}^{\mathbb{N}}$ and S are isomorphic

$\mathbb{R}^{\mathbb{N}}$ and S are really the same space in two different disguises.

$$f \in \mathbb{R}^{\mathbb{N}} \implies f \text{ determines an associated sequence } (f(1), f(2), \dots)$$

and

$$x = (x_1, x_2, \dots) \in S \implies x \text{ determines an associated function } f : \mathbb{N} \to \mathbb{R} \text{ by}$$

$$f(n) = x_n$$

Exercise

Make this precise as follows. Define

$$\begin{aligned} \varphi : \mathbb{R}^{\mathbb{N}} &\longrightarrow S \\ f &\longmapsto (f(1), f(2), \dots) \end{aligned}$$

That is, if $f : \mathbb{N} \to \mathbb{R}$ then $\varphi(f)$ is the sequence

$$\varphi(f) = (f(1), f(2), \dots)$$

Prove the following.

a. φ is a bijection

b. φ is linear, ie., it preserves the vector space operations: $\forall f, g \in \mathbb{R}^{\mathbb{N}}$ and $\forall c \in \mathbb{R}$,

$$\varphi(f+g) = \varphi(f) + \varphi(g)$$

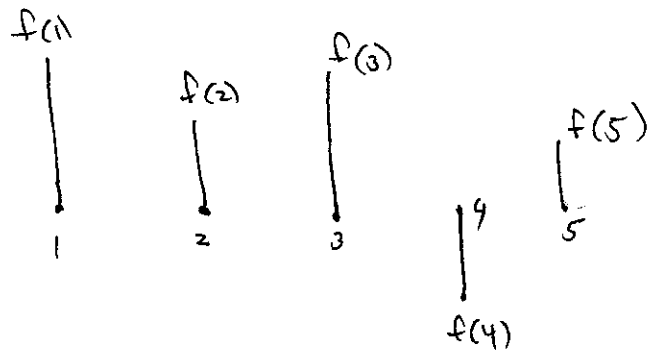
$$\varphi(cf) = c\varphi(f)$$

A linear bijection between vector spaces is called an isomophism. Thus $\varphi: \mathbb{R}^{\mathbb{N}} \to S$ is an isomorphism, and we say that $\mathbb{R}^{\mathbb{N}}$ and S are isomorphic.

$\mathbb{R}^{\{1,2,3,4,5\}}$

By definition,

$$\mathbb{R}^{\{1,2,3,4,5\}} = \left\{ f \;:\; f : \{1,2,3,4,5\} \to \mathbb{R} \right\}$$



In contrast, $\mathbb{R}^5$ is the space of ordered 5-tuples:

$$\mathbb{R}^5 = \left\{ x = (x_1, x_2, x_3, x_4, x_5) \;:\; x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

<u>Exercise</u>
Show $\mathbb{R}^{\{1,2,3,4,5\}}$ & $\mathbb{R}^5$ are isomorphic. What is the isomorphism?

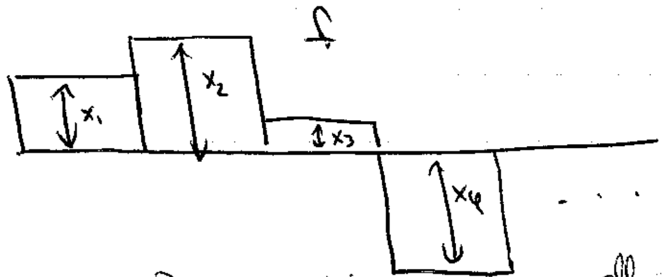
Thus, a sequence $(x_1, x_2, x_3, x_4, x_5)$ "really" is a function with domain $\{1,2,3,4,5\}$ and function values $f(1) = x_1$

$$f(2) = x_2$$
$$f(3) = x_3$$
$$f(4) = x_4$$
$$f(5) = x_5.$$

Sequences can be viewed as discrete functions

$(x_1, x_2, \ldots, x_n)$ represents $f : \{1, \ldots, n\} \to \mathbb{R}$

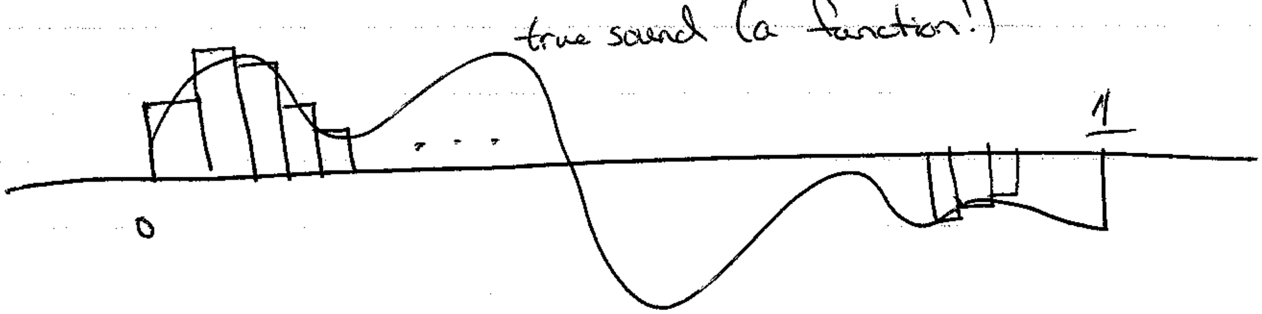
$f(1) = x_1$
$f(2) = x_2$
$\vdots$
$f(n) = x_n$



$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_k \in \mathbb{R}\} \xleftrightarrow[\text{correspondence}]{1\text{-}1} S = \{\text{all } f : \{1, \ldots, n\} \to \mathbb{R}\}$

More than just 1-1 corr: Addition, mult in $\mathbb{R}^n$ reflected in add., mult in S.

Ex. CD sound.



1 sec CD sound stored as sequence of 44 000 nos

$= $ one vector in $\mathbb{R}^{44{,}000}$

$= $ one function in $\mathbb{R}^{\{1, \ldots, 44{,}000\}}$.

The true 1 sec of CD sound can be viewed as a function f on $[0,1]$

$$f \in \mathbb{R}^{[0,1]}.$$

Norms

If V is a vector space then a function $\|\cdot\|: V \to \mathbb{R}$ is a norm on V if

a. $\|x\| \geq 0 \quad \forall x \in V$

b. $\|x\| = 0 \iff x = 0$

c. $\|cx\| = |c| \, \|x\| \quad \forall c \in \mathbb{R}, x \in V$

d. Triangle Inequality: $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

Example

The norm that will mostly concern us is the Euclidean norm on $\mathbb{R}^p$, which is defined by

$$\|x\| = \left(x_1^2 + \cdots + x_p^2 \right)^{1/2}, \qquad x \in \mathbb{R}^p.$$

However, there are infinitely many other norms on $\mathbb{R}^p$

If q is any real number in the range $1 \leq q < \infty$, then

$$\|x\|_q = \left(|x_1|^q + \cdots + |x_p|^q \right)^{1/q}, \qquad x \in \mathbb{R}^p$$

is a norm on $\mathbb{R}^p$. Another norm is

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_p|\}.$$

<u>Exercise:</u> Prove that $\|\cdot\|_q$ is a norm when $q=1$ or $q=\infty$ (those are the easy ones, the proof for other values of q is trickier).

<u>Example</u>

Let $S = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R}\}$ be the space of all infinite sequences of real numbers. Is

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|, \qquad x \in S$$

a norm on S?

The answer is no, because $\|x\|_1$ need not be finite, and by definition a norm on S must take only (nonnegative) <u>real</u> values.

One way to "fix" this is to consider a subspace of S: define

$$\ell^1 = \left\{ x = (x_1, x_2, \dots) \in S \ : \ \sum_{k=1}^{\infty} |x_k| < \infty \right\}$$

Examples: $(1,1,1,\dots) \notin \ell^1$

$(1, \frac{1}{2}, \frac{1}{3}, \dots) \notin \ell^1$

$(1, \frac{1}{4}, \frac{1}{9}, \dots) \in \ell^1$

<u>Exercise</u> ℓ^1 is a vector space, and

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|, \qquad x \in \ell^1$$

<u>is</u> a norm on ℓ^1.

Likewise, given any $1 \leq q < \infty$, define

$$\ell^q = \left\{ x = (x_1, x_2, \dots) \in S : \sum_{k=1}^{\infty} |x_k|^q < \infty \right\}$$

It is not so easy to prove when $q \neq 1$, but it is true that ℓ^q is a vector space, and

$$\|x\|_q = \left(\sum_{k=1}^{\infty} |x_k|^q \right)^{1/q}, \qquad x \in \ell^q$$

is a norm on ℓ^q.

<u>Exercise:</u> Show that

$$\ell^\infty = \left\{ x = (x_1, x_2, \dots) \in S : \sup_k |x_k| < \infty \right\}$$

is a vector space, and

$$\|x\|_\infty = \sup_k |x_k|, \quad x \in S$$

is a norm on ℓ^∞.

Show that if $1 < q < \infty$, then

$$\ell^1 \subsetneq \ell^q \subsetneq \ell^\infty.$$

Inner Products and Their Norms

Recall the dot product of vectors in $\mathbb{R}^p$:

$$x \cdot y = x_1 y_1 + \cdots + x_p y_p.$$

This is an example of an inner product.

Definition

An inner product on a vector space V is a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

that satisfies

a. $x \cdot x \geq 0$

b. $x \cdot x = 0 \iff x = 0$

c. $x \cdot y = y \cdot x$

d. $x \cdot (y + z) = x \cdot y + x \cdot z$

e. $(cx) \cdot y = c(x \cdot y)$.

Other notations

On spaces other than $\mathbb{R}^p$, we usually write $\langle x,y\rangle$ or $[x,y]$ instead of $x\cdot y$ for an inner product. Even on $\mathbb{R}^p$ we may use this notation if we are using an inner product different than the standard dot product.

Exercise

Let $\omega_1,\dots,\omega_p$ be fixed positive real numbers. Show that

$$\langle x,y\rangle = x_1y_1\omega_1 + \cdots + x_py_p\omega_p$$

is an inner product on $\mathbb{R}^p$.

Example

$$\langle x,y\rangle = \sum_{k=1}^{\infty} x_ky_k, \qquad x,y\in l^2$$

is an inner product on l^2.

<u>Claim</u>: All inner products lead to norms.
Not all norms lead to inner products.

<u>Theorem</u>

If V is an inner product space then $\|x\| = (x \cdot x)^{1/2}$ is a norm on V. Further,

$$x \cdot y \leq \|x\|\,\|y\| \quad \text{(Weak Schwarz Inequality)}$$

and $\forall x, y \neq 0$,
and $\|x \cdot y\| = \|x\|\,\|y\|$ if & only if $x = cy$ for some $c > 0$.

<u>Proof</u>;

(i) $\|x\| = (x \cdot x)^{1/2} \geq 0$ ✓

(ii) $\|x\| = (x \cdot x)^{1/2} = 0 \iff x \cdot x = 0 \iff x = 0$ ✓

(iii) $\|ax\|^2 = (ax \cdot ax) = a\,(x \cdot ax) = a^2\,(x \cdot x)$
So $\|ax\| = |a|\,\|x\|$.

(iv) Set $z = ax - by$. Then

$$\begin{aligned} 0 \leq z \cdot z &= (ax - by) \cdot (ax - by) \\ &= a^2\, x \cdot x - 2ab\, x \cdot y + b^2\, y \cdot y \\ &= a^2 \|x\|^2 - 2ab\, x \cdot y + b^2 \|y\|^2 \end{aligned}$$

~~Set~~ For the specific case of $a = \|y\|$ & $b = \|x\|$, ~~Then~~ this says that

$$0 \le \|y\|^2 \|x\|^2 - 2\|y\|\,\|x\|\, x\cdot y + \|x\|^2 \|y\|^2$$

$$= 2\|x\|\,\|y\| \left(\|x\|\,\|y\| - x\cdot y \right).$$

Since $\|x\|, \|y\| \ge 0$, we have $\|x\|\,\|y\| - x\cdot y \ge 0$.

Hence $x\cdot y \le \|x\|\,\|y\|$ is true. Therefore

$$\|x+y\|^2 = (x+y)\cdot(x\cdot y)$$

$$= x\cdot x + 2x\cdot y + y\cdot y$$

$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= \left(\|x\| + \|y\|\right)^2,$$

so $\|x+y\| \le \|x\| + \|y\|$.

<u>Exercise</u>: Show $x\cdot y = \|x\|\,\|y\| \iff x = cy$ with $c > 0$.

Thus every inner product has an associated norm.

Corollary

$$|x \cdot y| \le \|x\| \, \|y\| \qquad \text{(Schwarz Inequality)}$$

And, $\forall y \neq 0$, $\quad |x \cdot y| = \|x\| \, \|y\| \iff x = ay$.

Not all norms lead to inner products!

Ex: There is no inner product on $\mathbb{R}^p$ s.t.

$$\|x\|_1 = (x \cdot x)^{1/2}$$

But, $\mathbb{R}^p$ does have some norm that comes from an inner product ($\mathbb{R}^p$ is a Hilbert space)

Exercise
$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

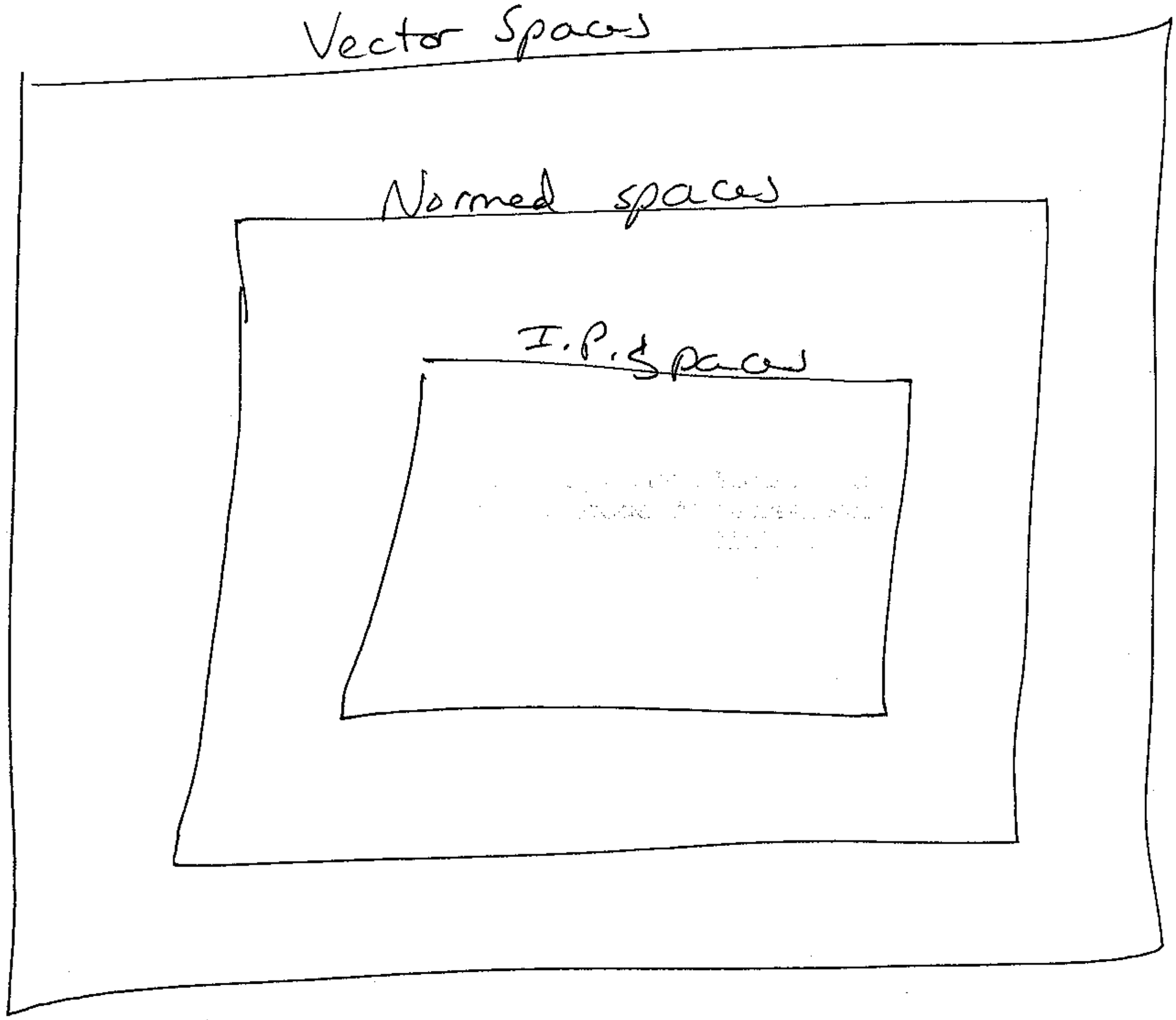
Ex: There is no norm on ℓ^1 that comes from an inner product.

Fails parallelogram law

Contrast $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$ on ℓ^2

does come from an inner product.

V i.p. space w/ i.p. $x \cdot y$ $\Longrightarrow$ $\|x\| = (x \cdot x)^{1/2}$ is a norm



Same space can have <u>many</u> norms or ips.

We'll mostly use the
<u>Euclidean norm on $\mathbb{R}^p$</u>

<u>NOTE</u> $\|x\| = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_p^2}$

$\boxed{\text{Unless otherwise specified, this is the norm we mean.}}$ $\Leftarrow$

Open ball:

Notation Alert!

$$B_r(x) = \{y \in \mathbb{R}^p : \|x-y\| < r\}$$

← NOT Bartle notation !!

Closed ball

$$\overline{B_r}(x) = \{y \in \mathbb{R}^p : \|x-y\| \leq r\}.$$

<u>Sphere:</u>

$$\overline{B_r}(x) \setminus B_r(x) = \{y \in \mathbb{R}^p : \|x-y\| = r\}.$$

<u>Note:</u> $p=1$ balls are intervals

$$B_r(x) = (x-r, x+r)$$