1. Given \( f \in L^1[0,1] \) (under Lebesgue measure), define \( g(x) = \int_x^1 f(t) \frac{dt}{t} \). Show that \( g \) is defined a.e., that \( g \in L^1[0,1] \), and that \( \int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx \).

2. Let \((X, \mathcal{M}, \mu)\) be a measure space such that \( \mu(X) < \infty \). Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of measurable functions, and let \( f \) be a measurable function. Prove that \( f_n \overset{m}{\to} f \iff \lim_{n \to \infty} \int_X \frac{|f - f_n|}{1 + |f - f_n|} \, d\mu = 0 \).

3. Let \((X, \mathcal{M}, \mu)\) be a measure space such that \( \mu(X) < \infty \). Suppose that \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of measurable functions that are finite a.e., and \( f \) is a measurable function such that \( f_n(x) \to f(x) \) pointwise a.e. Prove that there exist disjoint measurable sets \( E_0, E_1, \ldots \) with \( \bigcup E_k = X \) such that \( \mu(E_0) = 0 \), and for each fixed \( k \in \mathbb{N} \) we have that \( f_n \to f \) uniformly on \( E_k \) as \( n \to \infty \).

4. Scalars in this problem are complex. Let \( f, g \in L^1(\mathbb{R}) \) be given (Lebesgue measure).

(a) Prove that \((f * g)(x) = \int f(y) g(x - y) \, dy\) is measurable and \( f * g \in L^1(\mathbb{R}) \). You may assume without proof that \( f(y) g(x - y) \) is measurable on \( \mathbb{R}^2 \).

(b) We proved in Homework 6 that \( \hat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} \, dx \) is defined for every \( \xi \), and that \( \hat{f} \) is a continuous and bounded function. Prove that \((f * g)\wedge(\xi) = \hat{f}(\xi) \hat{g}(\xi), \quad \xi \in \mathbb{R} \).

5. Suppose that \( f: [0,1] \to [0,\infty] \) belongs to \( L^1[0,1] \) (Lebesgue measure) and that for each \( n \in \mathbb{N} \) we have

\[
\int_0^1 f(x)^n \, dx = \int_0^1 f(x) \, dx.
\]

Prove that there exists a measurable set \( E \subseteq [0,1] \) such that \( f = \chi_E \) a.e.