MATH 6337 FINAL EXAM December 12, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (15 points) and choose THREE other problems to answer (15 points each). You may also answer (for up to 5 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem.

All statements require proof or justification. There are 60 points total, plus up to 5 points of extra credit.

Throughout this exam, you should assume that scalars are REAL unless specified otherwise.

1. The two parts of this problem are not related.

(a) Let $g\colon [a,b] \to \mathbb{R}$ be given. Prove that g is Lipschitz if and only if g is absolutely continuous and there exists $M > 0$ such that $|g'(x)| \le M$ a.e.

(b) Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ has the property that

$$\forall\, k \in \mathbb{N}, \quad \forall\, j \in \mathbb{Z}, \quad \int_{j/2^k}^{(j+1)/2^k} f(t)\, dt \;=\; 0.$$

In other words, $\int_j^{j+1} f = 0$ for all j, $\int_{j/2}^{(j+1)/2} f = 0$ for all j, $\int_{j/4}^{(j+1)/4} f = 0$ for all j, etc. Prove that $f = 0$ a.e.

2. (a) Let μ, ν be σ-finite positive measures on $(X, \mathcal{M})$. Suppose that $\nu \ll \mu$ and that the Radon–Nikodym derivative $f = d\nu/d\mu$ satisfies

$$f(x) \;=\; \frac{d\nu}{d\mu}(x) \;\in\; (0,\infty), \qquad \text{all } x \in X.$$

Show that $\mu \ll \nu$, and find a formula for $d\mu/d\nu$ (in terms of $d\nu/d\mu$).

(b) Now suppose that μ, ν, λ are σ-finite positive measures on $(X, \mathcal{M})$, and that $\lambda \ll \mu, \nu$, with $\frac{d\lambda}{d\mu}(x)$, $\frac{d\lambda}{d\nu}(x) \in (0,\infty)$ for every x. Find a formula for

$$\frac{d\lambda}{d(\mu+\nu)} \quad \text{in terms of } \frac{d\lambda}{d\mu} \text{ and } \frac{d\lambda}{d\nu}.$$

In other words, let $d\lambda = f\, d\mu = g\, d\nu$, and show that $d(\mu + \nu) = h\, d\lambda$, where h is expressed in terms of f and g alone.

3. Let $(X, \mathcal{M}, \mu)$ be a complete measure space (so $\mu \ge 0$). Let $L^1(X)$ denote the space of all integrable functions $f\colon \mathbb{R} \to \overline{\mathbb{R}}$, with norm

$$\|f\|_1 \;=\; \int_X |f(x)|\, d\mu(x).$$

Show that if we identify functions that are equal almost everywhere, then $L^1(X)$ is a Banach space with respect to the norm $\|\cdot\|_1$. In other words, show that if $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_1$, then there exists $f \in L^1(\mathbb{R})$ such that $\|f - f_n\|_1 \to 0$ as $N \to \infty$.

4. The two parts of this problem are not related.

(a) Let $(X, \mathcal{M}, \mu)$ be a measure space (so $\mu \geq 0$), and suppose that $f \in L^1(\mu)$ with $f(x) \geq 0$ for all $x \in X$. Prove that

$$\lim_{n\to\infty} \int n \ln\Bigl(1 + \frac{f(x)}{n}\Bigr)\, d\mu(x) \;=\; \int f(x)\, d\mu(x).$$

(b) Let $f\colon [0,1] \to \mathbb{R}$ be Lebesgue measurable. Show that if $h(x,y) = f(x) - f(y)$ is integrable on $[0,1] \times [0,1]$ then $f \in L^1[0,1]$.

5. The two parts of this problem are not related. Scalars in this problem are *complex.*

(a) Let ν be a complex measure on a measurable space $(X, \mathcal{M})$. Show that if $\nu(X) = |\nu|(X)$, then ν is a positive measure.

(b) Let ν be a complex measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Show that if $E \subseteq \mathbb{N}$, then

$$|\nu|(E) \;=\; \sum_{k\in E} |\nu\{k\}|.$$

6. (a) Briefly (one paragraph) explain how a Lebesgue–Stieltjes measure on $\mathbb{R}$ is constructed, and state any one "regularity" property of such a measure (proof not required).

(b) Let F be the Cantor–Lebesgue function on $[0,1]$. Extend F to all of $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Explain briefly why $F \in \mathrm{NBV}(\mathbb{R})$.

(c) Let μ_F be the corresponding Lebesgue–Stieltjes measure. Give a direct proof that $\mu_F \perp dx$.

(d) By appealing to theorems from class, give another proof that $\mu_F \perp dx$.

7. The two parts of this problem are not related.

(a) Let $(X, \mathcal{M}, \mu)$ be a finite measure space (so $\mu \geq 0$), and let f_n, $f\colon X \to \mathbb{R}$ be measurable functions. Show that if $f_n(x) \to f(x)$ pointwise for a.e. x, then $f_n \xrightarrow{\mathrm{m}} f$ (convergence in measure).

(b) Recall that the convolution of f, $g \in L^1(\mathbb{R})$ is the function $f * g$ given by

$$(f * g)(x) \;=\; \int f(y)\, g(x-y)\, dy,$$

defined for all x where the integral exists. Prove that $(f * g)(x)$ exists for a.e. x, that $f * g$ is measurable, and that $f * g \in L^1(\mathbb{R})$.