REAL ANALYSIS LECTURE NOTES:

SHORT REVIEW OF METRICS, NORMS, AND CONVERGENCE

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In these notes we will give some review of basic notions and terminology for metrics, norms, and convergence. Expanded coverage of much of this material can be found in the supplementary notes posted at the class website.

1. Metrics and Convergence

A metric determines a notion of distance between points in a set.

Definition 1 (Metric Space)**.** Let X be a set. A *metric* on X is a function $\mathrm{d}\colon X \times X \to \mathbb{R}$ such that:

(a) $\mathrm{d}(f,g) \geq 0$ for all $f,\, g \in X$,

(b) $\mathrm{d}(f,g) = 0$ if and only if $f = g$,

(c) $\mathrm{d}(f,g) = \mathrm{d}(g,f)$ for all $f,\, g \in X$,

(d) Triangle Inequality: for all $f,\, g,\, h \in X$ we have

$$\mathrm{d}(f,h) \;\leq\; \mathrm{d}(f,g) + \mathrm{d}(g,h).$$

In this case, X is a called a *metric space.* The value $\mathrm{d}(f,g)$ is the *distance* from f to g.

A metric space need not be a vector space, although this will be true of most of the metric spaces we will encounter.

Once we have a notion of distance, we have a corresponding notion of convergence.

Definition 2 (Convergent and Cauchy sequences)**.** Let X be a metric space with metric d, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

(a) We say that $\{f_n\}_{n\in\mathbb{N}}$ *converges* to $f \in X$ if $\lim\limits_{n\to\infty} \mathrm{d}(f_n, f) = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, n \geq N, \quad \mathrm{d}(f_n, f) < \varepsilon.$$

In this case, we write $\lim\limits_{n\to\infty} f_n = f$ or $f_n \to f$.

(b) We say that $\{f_n\}_{n\in\mathbb{N}}$ is *Cauchy* if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, m, n \geq N, \quad \mathrm{d}(f_m, f_n) < \varepsilon.$$

Exercise 3. Let X be a metric space.

(a) Every convergent sequence in X is Cauchy.

(b) The limit of a convergent sequence is unique.

In general, however, a Cauchy sequence need not be convergent (see Exercise 11).

Definition 4 (Complete Metric Space)**.** If every Cauchy sequence in a metric space X has the property that it converges to an element of X, then X is said to be *complete*.

Beware that the term "complete" is heavily overused and has a number of distinct mathematical meanings (for example, we have seen the terminology "complete measure space", and another completely distinct notion is the definition of a "complete sequence" of vectors in a Banach space).

Actually, for finite measure spaces the definition of convergence in measure can be reformulated in terms of a metric.

Exercise 5. Suppose that μ is a finite measure, and let

$$V = \{f \colon X \to \mathbb{C} : f \text{ is measurable}\}.$$

For f, $g \in V$, define

$$\mathrm{d}(f,g) = \int \frac{|f-g|}{1+|f-g|}.$$

Show that d is a metric on V if we identify functions that are equal a.e., and show that

$$f_n \xrightarrow{\mathrm{m}} f \quad \Longleftrightarrow \quad \mathrm{d}(f_n, f) \to 0.$$

Hint: The function $\frac{x}{x+1}$ is an increasing function of x.

2. Norms and Seminorms

A norm provides a notion of the length of a vector in a vector space. In these notes, we will take our vector spaces to be over the complex field $\mathbb{C}$, but only minor changes are needed if we instead assume that they are over the real field $\mathbb{R}$.

Definition 6 (Seminorms and Norms)**.** Let X be a vector space over the field $\mathbb{C}$ of complex scalars. A *seminorm* on X is a function $\|\cdot\| \colon X \to \mathbb{R}$ such that for all f, $g \in X$ and all scalars $c \in \mathbb{C}$ we have:

(a) $\|f\| \geq 0$,

(b) $\|cf\| = |c|\,\|f\|$, and

(c) Triangle Inequality: $\|f+g\| \leq \|f\| + \|g\|$.

A seminorm is a *norm* if we also have:

(d) $\|f\| = 0$ if and only if $f = 0$.

A vector space X together with a norm $\|\cdot\|$ is called a *normed linear space* or simply a *normed space*.

Note that if S is a subspace of a normed space X, then S is itself a normed space with respect to the norm on X (restricted to S).

The following exercise shows that all normed spaces are metric spaces. In particular, the notions of convergent and Cauchy sequences apply in any normed space.

Exercise 7. If X is a normed space, then

$$\mathrm{d}(f,g) \;=\; \|f-g\|$$

defines a metric on X, called the *induced metric*.

Not every metric is induced from a norm; the metric in Exercise 5 corresponding to convergence in measure is an example.

Exercise 8. Show that if X is a normed linear space, then the following statements hold.

(a) Reverse Triangle Inequality: $\big|\|f\| - \|g\|\big| \le \|f-g\|$.

(b) Continuity of the norm: $f_n \to f \implies \|f_n\| \to \|f\|$.

(c) Continuity of vector addition: $f_n \to f$ and $g_n \to g \implies f_n + g_n \to f+g$.

(d) Continuity of scalar multiplication: $f_n \to f$ and $\alpha_n \to \alpha \implies \alpha_n f_n \to \alpha f$.

(e) Boundedness of convergent sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is convergent then $\sup \|f_n\| < \infty$.

(f) Boundedness of Cauchy sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy then $\sup \|f_n\| < \infty$.

Definition 9 (Banach Space)**.** A normed linear space X is called a *Banach space* if it is complete under the induced metric, i.e., if every Cauchy sequence is convergent.

Thus, the terms "Banach space" and "complete normed space" are interchangeable.

An important fact that we will assume without proof is that the complex plane $\mathbb{C}$ under absolute value is a Banach space.

3. Examples of Banach Spaces: ℓ^p

In this section we give a few examples of Banach spaces and complete metric spaces.

We begin with the ℓ^p spaces on countable index sets.

Definition 10. Let I be a finite or countably infinite index sequence.

(a) If $1 \le p < \infty$, then $\ell^p(I)$ consists of all sequences of scalars $x = (x_k)_{k\in I}$ such that

$$\|x\|_p \;=\; \|(x_k)_{k\in I}\|_p \;=\; \left(\sum_{k\in I} |x_k|^p\right)^{1/p} \;<\; \infty.$$

(b) For $p=\infty$, the space $\ell^\infty(I)$ consists of all sequences of scalars $x=(x_k)_{k\in I}$ such that

$$\|x\|_\infty \;=\; \|(x_k)_{k\in I}\|_\infty \;=\; \sup_{k\in I} |x_k| \;<\; \infty.$$

If $I = \mathbb{N}$, then we write simply ℓ^p instead of $\ell^p(\mathbb{N})$.

If $I = \{1,\dots,d\}$, then $\ell^p(I) = \mathbb{C}^d$, and in this case we refer to $\ell^p(I)$ as "$\mathbb{C}^d$ under the ℓ^p norm." The ℓ^2 norm on $\mathbb{C}^d$ is called the *Euclidean norm.*

It is a fact that each ℓ^p space for $1 \le p \le \infty$ is a Banach space. However, it is not so easy to prove the Triangle Inequality when $1 < p < \infty$ — for this you need a result called *Hölder's Inequality* that we will cover later. On the other hand, if we assume this is true, then we can prove that ℓ^p is complete.

Exercise 11. (a) Prove that $\|\cdot\|_1$ is a norm on ℓ^1, and that $\|\cdot\|_\infty$ is a norm on ℓ^∞. Show that $\ell^\infty \subsetneq \ell^1$.

(b) With $1 \le p \le \infty$ fixed, prove that ℓ^p is complete.

Hints: Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^p, and write $x_n = (x_n(1), x_n(2), \dots)$. Then show that for each *fixed* k we have that $\{x_n(k)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars, hence converges, say $y_k = \lim_{k\to\infty} x_n(k)$. Thus, for each k, the kth component of x_n converges to the kth component of y; this is called *componentwise convergence*.

Now we have a candidate sequence $y = (y_1, y_2, \dots)$ for the limit of $\{x_n\}_{n\in\mathbb{N}}$. Use the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy together with the componentwise convergence to show that $\|x - x_n\|_p \to 0$.

(c) Define

$$c_{00} \;=\; \big\{x = (x_1, \dots, x_N, 0, 0, \dots) : N > 0,\, x_1, \dots, x_N \in \mathbb{C}\big\},$$

and observe that $c_{00} \subsetneq \ell^p$ for every p. The vectors in c_{00} are sometimes called *finite sequences* because they contain at most finitely many nonzero components.

Let us fix $p = 1$. The space c_{00} is a normed space if we restrict $\|\cdot\|_1$ to it. Find a sequence of vectors $\{x_n\}_{n\in\mathbb{N}}$ in c_{00} that is Cauchy with respect to this norm, but does not converge *in* c_{00} with respect to this norm.

Hint: Since $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in ℓ^1, it must converge to some vector *in* ℓ^1. Try to arrange it so that the limit vector x does not belong to c_{00}.

4. Examples of Banach Spaces: C_b, C_0, C_b^m

Here are some additional examples of Banach spaces of functions on $\mathbb{R}$. The *support* of a function is the closure of the set where it is nonzero, i.e.,

$$\operatorname{supp}(f) \;=\; \overline{\{x \in \mathbb{R} : f(x) \ne 0\}}.$$

Since $\operatorname{supp}(f)$ is closed, it is compact if and only if it is bounded. Thus, a function has *compact support* if and only if it is zero outside of some finite interval.

Exercise 12. (a) Let $C_b(\mathbb{R})$ denote the space of continuous, bounded functions $f\colon \mathbb{R} \to \mathbb{C}$. Show that $C_b(\mathbb{R})$ is a Banach space with respect to the *uniform norm*

$$\|f\|_\infty \;=\; \sup_{t\in\mathbb{R}} |f(t)|.$$

(b) Show that the subspace

$$C_0(\mathbb{R}) \;=\; \big\{f \in C_b(\mathbb{R}) : \lim_{|t|\to\infty} f(t) = 0\big\}$$

is also a Banach space with respect to the uniform norm.

(c) Show that the subspace

$$C_c(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact}\bigr\} \tag{1}$$

is a normed space that is not a Banach space under the uniform norm.

Hint: Use a similar idea as in Exercise 11(c): Find functions $f_n \in C_c(\mathbb{R})$ that converge to a function $f \in C_0(\mathbb{R})$ with respect to the uniform norm, but $f \notin C_c(\mathbb{R})$.

Beware, some authors use the symbols C_0 to denote the space that we call C_c.

Exercise 13. Let $C_b^m(\mathbb{R})$ be the space of all m-times differentiable functions on $\mathbb{R}$ each of whose derivatives is bounded and continuous, i.e.,

$$C_b^m(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : f, f', \dots, f^{(m)} \in C_b(\mathbb{R})\bigr\}.$$

Show that $C_b^m(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{C_b^m} \;=\; \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(m)}\|_\infty.$$

5. Closed and Dense Subspaces

Definition 14. Let X be a normed linear space.

(a) A subspace $M \subseteq X$ is *closed* if it contains all its limit points, i.e., if

$$x_n \in M \text{ and } x_n \to x \in X \quad\Longrightarrow\quad x \in M.$$

(b) A subspace $S \subseteq X$ is *dense* if every element of element is a limit point of S, i.e., if

$$x \in X \quad\Longrightarrow\quad \exists\, x_n \in S \text{ such that } x_n \to x.$$

In finite-dimensional spaces, there are no proper dense subspaces. For example, the rationals $\mathbb{Q}$ are a dense *subset* of $\mathbb{R}$, but they're not a *subspace* (because $\mathbb{Q}$ is not closed under multiplication by all real scalars). But in infinite-dimensional spaces we can give many examples.

Exercise 15. Let M be a subspace of a Banach space X. Show that M is a normed space using the norm of X restricted to M, and show that

$$M \text{ is a Banach space} \qquad\Longleftrightarrow\qquad M \text{ is closed.}$$

Here, we are using the norm on M that is inherited from X — we cannot say what will happen if we put a different norm on M.

Exercise 16. (a) Fix $1 \le p \le \infty$. Prove that the space c_{00} introduced in Exercise 11 is a subspace of ℓ^p that is not closed (with respect to the ℓ^p-norm). Prove that c_{00} is dense in $\ell^p(\mathbb{N})$ if $p < \infty$, but that it is not dense in ℓ^∞.

(b) Define

$$c_0 \;=\; \Bigl\{x = (x_k)_{k=1}^\infty : \lim_{k\to\infty} x_k = 0\Bigr\}.$$

Prove c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$, and that c_{00} is dense in c_0 with respect to the ℓ^∞-norm.

Exercise 17. Show that the space $C_c(\mathbb{R})$ introduced in equation (1) is a dense subspace of $C_0(\mathbb{R})$ that is not closed (under the uniform norm).