# REAL ANALYSIS LECTURE NOTES:

# 1.4 OUTER MEASURE

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## 1.4.1 Introduction

We will expand on Section 1.4 of Folland's text, which covers abstract outer measures also called *exterior measures*). To motivate the general theory, we incorporate material from Chapter 3 of Wheeden and Zygmund's text, in order to construct the fabled Lebesgue measure on $\mathbb{R}^d$. We then use this to consider abstract outer measures.

The steps in the construction of Lebesgue measure are as follows.

(a) We start with a basic class of subsets of $\mathbb{R}^d$ that we know how to measure, namely, cubes or rectangular boxes. We declare that their measure is their volume.

(b) We next find a way to extend the notion of measure to *all* subsets of $\mathbb{R}^d$. For every $E \subseteq \mathbb{R}^d$ we define a nonnegative, extended real-value number that we call $\mu^*(E)$ or $|E|_e$ in a way that naturally extends the notion of the volume of cubes. This is *exterior Lebesgue measure*. The good news is that every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure. The bad news is that $\mu^*$ is *not* a measure — it is countably subadditive but not countably additive.

(c) Finally, we find a way to restrict our attention to a smaller class of sets, the $\sigma$-algebra $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbb{R}^d$. We show that if $\mu$ is $\mu^*$ restricted to *this* $\sigma$-algebra, then $\mu$ is indeed a measure. *This* measure is Lebesgue measure on $\mathbb{R}^d$.

The same idea applies more generally: If we begin with some class of subsets of $X$ that already have an assigned measure, then perhaps we can extend to a subadditive outer measure that is defined on all subsets of $X$, and then by restricting to some appropriate smaller $\sigma$-algebra obtain a measure on $X$. But first we will begin by constructing exterior Lebesgue measure on $\mathbb{R}^d$.

## 1.4.2 Exterior Lebesgue Measure

We begin with the familiar notion of the volume of a rectangular box in $\mathbb{R}^d$, which for simplicity we refer to as a "cube" (even though we do not require all side lengths to be equal). Other common names for such sets are *intervals*, *rectangles*, or *rectangular boxes*.

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These notes follow and expand on the text "Real Analysis: Modern Techniques and their Applications," 2nd ed., by G. Folland. The material on Lebesgue measure is based on the text "Measure and Integral," by R. L. Wheeden and A. Zygmund.

**Definition 1.** A *cube* in $\mathbb{R}^d$ is a set of the form

$$Q \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d] \;=\; \prod_{i=1}^{d} [a_i, b_i].$$

The *volume* of this cube is

$$\mathrm{vol}(Q) \;=\; (b_1 - a_1) \cdots (b_d - a_d) \;=\; \prod_{i=1}^{d} (b_i - a_i).$$

We extend the notion of volume to arbitrary sets by covering them with countably many cubes in all possible ways (for us, a countable set means either a finite set or a countably infinite set, although sometimes we repeat the phrase "finite or countably infinite" for emphasis). For simplicity of notation, we will write $\{Q_k\}_k$ to denote a collection of cubes $Q_k$, with $k$ running through some implicit index set (usually either finite or countable). Alternatively, if we declare that the empty set is also a cube, then we can always consider a finite collection $\{Q_k\}_{k=1}^N$ to be an infinite collection $\{Q_k\}_{k=1}^\infty$ where $Q_k = \emptyset$ for $k > N$.

**Definition 2.** The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e \;=\; \inf\Big\{\sum_k \mathrm{vol}(Q_k)\Big\}$$

where the infimum is taken over all all *finite or countably infinite* collections of cubes $Q_k$ such that $E \subseteq \cup Q_k$.

We prefer to denote exterior Lebesgue measure by $|E|_e$, though if we wanted to be more consistent with Folland's notation, we should write $\mu^*(E)$ instead.

Every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure, which lies in the range

$$0 \;\le\; |E|_e \;\le\; \infty.$$

Note the following immediate, but important, consequences of the definition of exterior measure.

- If $Q_k$ are countably many cubes and $E \subseteq \cup Q_k$, then $|E|_e \le \sum \mathrm{vol}(Q_k)$.
- Given $\varepsilon > 0$, there exist countably many cubes $Q_k$ with $E \subseteq \cup Q_k$ such that

  $$|E|_e \;\le\; \sum_k \mathrm{vol}(Q_k) \;\le\; |E|_e + \varepsilon.$$

  Note that we might have $|E|_e = \infty$ in the line above.

**Example 3.** Suppose that $E = \{x_1, x_2, \dots\}$ is a countable subset of $\mathbb{R}^d$, and choose any $\varepsilon > 0$. For each $k$, choose a cube $Q_k$ that contains $x_k$ and that has volume $\mathrm{vol}(Q_k) < \varepsilon/2^k$. Then $E \subseteq \cup Q_k$, so $|E|_e \le \sum \mathrm{vol}(Q_k) \le \varepsilon$. Since $\varepsilon$ is arbitrary, we conclude that $|E|_e = 0$. Thus, every countable subset of $\mathbb{R}^d$ has exterior measure zero.

Next we will explore some of the properties of Lebesgue measure.

**Lemma 4** (Monotonicity)**.** If $A \subseteq B \subseteq \mathbb{R}^d$, then $|A|_e \le |B|_e$.

*Proof.* If $\{Q_k\}_k$ is any countable cover of $B$ by cubes, then it is also a cover of $A$ by cubes, so we have

$$|A|_e \;\le\; \sum_k \mathrm{vol}(Q_k).$$

This is true for every possible covering of $B$, so

$$|A|_e \;\le\; \inf\Bigl\{\sum_k \mathrm{vol}(Q_k) :\; \text{all covers of } B \text{ by cubes}\Bigr\} \;=\; |B|_e. \qquad \square$$

The important point in proof is that if $\mathcal{C}_A$ is the collection of all covers of $A$ by cubes, and $\mathcal{C}_B$ the collection of covers of $B$, then $\mathcal{C}_B \subseteq \mathcal{C}_A$. Every covering of $B$ is a covering of $A$, but in general there are more ways to cover $A$ than there are to cover $B$.

**Exercise 5.** Let $C \subseteq [0,1] \subseteq \mathbb{R}$ be the classical middle-thirds Cantor set. Use monotonicity to show that the exterior Lebesgue measure of $C$ is $|C|_e = 0$. Thus $C$ is an uncountable subset of $\mathbb{R}$ that has exterior measure zero.

**Exercise 6** (Translation Invariance)**.** Show that if $E \subseteq \mathbb{R}$ and $h \in \mathbb{R}^d$, then $|E+h|_e = |E|_e$, where $E + h = \{x + h : x \in E\}$.

Note that if $Q$ is a cube, then the collection $\{Q\}$ containing only the single cube $Q$ is a covering of $Q$ by cubes. Hence we certainly have $|Q|_e \le \mathrm{vol}(Q)$. However, it requires some care to show that the exterior measure of a cube actually coincides with its volume.

**Theorem 7** (Consistency with Volume)**.** If $Q$ is a cube in $\mathbb{R}^d$ then $|Q|_e = \mathrm{vol}(Q)$.

*Proof.* We have seen that $|Q|_e \le \mathrm{vol}(Q)$, so we must prove the opposite inequality.

Let $\{Q_k\}_k$ be any countable covering of $Q$ by cubes, and fix any $\varepsilon > 0$. For each $k$, let $Q_k^*$ be any cube such that:

- $Q_k$ is contained in the interior of $Q_k^*$, i.e., $Q_k \subseteq (Q_k^*)^\circ$, and
- $\mathrm{vol}(Q_k^*) \le (1+\varepsilon)\,\mathrm{vol}(Q_k)$.

Then we have

$$Q \;\subseteq\; \bigcup_k Q_k \;\subseteq\; \bigcup_k (Q_k^*)^\circ.$$

Hence $\{(Q_k^*)^\circ\}$ is a countable open cover of the compact set $Q$. It must therefore have a finite subcover, i.e., there must exist an $N > 0$ such that

$$Q \subseteq \bigcup_{k=1}^N (Q_k^*)^\circ \subseteq \bigcup_{k=1}^N Q_k^*.$$

Now, since we are dealing with cubes, we have (proof by exercise) that

$$\mathrm{vol}(Q) \;\le\; \sum_{k=1}^N \mathrm{vol}(Q_k^*) \;\le\; (1+\varepsilon)\sum_{k=1}^N \mathrm{vol}(Q_k) \;\le\; (1+\varepsilon)\sum_k \mathrm{vol}(Q_k).$$

Since this is true for every covering $\{Q_k\}_k$, we conclude that

$$\mathrm{vol}(Q) \;\le\; (1+\varepsilon)\, \inf\Bigl\{\sum_k \mathrm{vol}(Q_k)\Bigr\} \;=\; (1+\varepsilon)\, |Q|_e.$$

Since $\varepsilon$ is arbitrary, it follows that $\mathrm{vol}(Q) \le |Q|_e$. □

We have constructed a function $|\cdot|_e$ that is defined on *every* subset of $\mathbb{R}^d$ and has the properties:

- $0 \le |E|_e \le \infty$ for every $E \subseteq \mathbb{R}^d$,
- $|E+h|_e = |E|$ for every $E \subseteq \mathbb{R}^d$ and $h \in \mathbb{R}^d$,
- $|Q|_e = \mathrm{vol}(Q)$ for every cube $Q$.

Looking back to our very first theorem in Section 1.1, it follows that $|\cdot|_e$ cannot possibly be countably additive, and therefore it *cannot* be a measure on $\mathbb{R}^d$:

*Exterior Lebesgue measure is not a measure on* $\mathbb{R}^d$.

Rather unsettlingly, there exist sets $E$, $F \subset \mathbb{R}^d$ such that

$$E \cap F = \emptyset \qquad \text{yet} \qquad |E \cup F|_e < |E|_e + |F|_e.$$

On the other hand, let us prove that exterior Lebesgue measure is at least countably subadditive.

**Theorem 8** (Countable Subadditivity)**.** If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then

$$\Bigl|\bigcup_k E_k\Bigr|_e \;\le\; \sum_k |E_k|_e.$$

*Proof.* If any $E_k$ satisfies $|E_k|_e = \infty$ then we are done, so let us assume that $|E_k|_e < \infty$ for every $k$. Fix any $\varepsilon > 0$. Then for each $k$ we can find a covering $\{Q_j^{(k)}\}_j$ of $E_k$ by cubes $Q_j^{(k)}$ in such a way that

$$\sum_j \mathrm{vol}(Q_j^{(k)}) \;\le\; |E_k|_e + \frac{\varepsilon}{2^k}.$$

Then we have

$$\bigcup_k E_k \;\subseteq\; \bigcup_{k,j} Q_j^{(k)},$$

so

$$\Bigl|\bigcup_k E_k\Bigr|_e \;\le\; \sum_{k,j} \mathrm{vol}(Q_j^{(k)}) \;\le\; \sum_k \Bigl(|E_k|_e + \frac{\varepsilon}{2^k}\Bigr) \;=\; \sum_k |E_k|_e \;+\; \varepsilon.$$

Since $\varepsilon$ is arbitrary, the result follows. □

The next result states that every set $E$ can be surrounded by an *open* set $U$ whose exterior measure is only $\varepsilon$ larger than that of $E$ (by monotonicity we of course also have $|E|_e \le |U|_e$, so the measure of $U$ is very close to the measure of $E$).

**Theorem 9.** If $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, then there exists an open set $U \supseteq E$ such that

$$|U|_e \ \le\ |E|_e + \varepsilon,$$

and hence

$$|E|_e \ =\ \inf\bigl\{|U|_e : U \text{ open}, U \supseteq E\bigr\}. \tag{1}$$

*Proof.* If $|E|_e = \infty$, take $U = \mathbb{R}^d$. Otherwise we have $|E|_e < \infty$, so by definition of exterior measure there must exist cubes $Q_k$ such that $E \subseteq \cup Q_k$ and $\sum \mathrm{vol}(Q_k) \le |E| + \frac{\varepsilon}{2}$. Let $Q_k^*$ be a larger cube that contains $Q_k$ in its interior, and such that $\mathrm{vol}(Q_k^*) \le \mathrm{vol}(Q_k) + 2^{-k-1}\varepsilon$. Let $U$ be the union of the interiors of the cubes $Q_k^*$. Then $E \subseteq U$, $U$ is open, and

$$|U|_e \ \le\ \sum_k \mathrm{vol}(Q_k^*) \ \le\ \sum_k \mathrm{vol}(Q_k) \ +\ \frac{\varepsilon}{2} \ \le\ |E| + \varepsilon. \qquad \square$$

Since $E$ and $U\backslash E$ are disjoint and their union is $U$, we might expect that the sum of their exterior measures is the exterior measure of $U$. Unfortunately, this is false in general (although the Axiom of Choice is required to show the existence of a counterexample). Consequently, the fact that $|U|_e \le |E|_e + \varepsilon$ does *not* imply that $|U\backslash E|_e \le \varepsilon$! The "well-behaved" sets for which this is true will be said to be *measurable*, and are studied in the next section.

The next exercise pushes this "surrounding" issue a bit further.

**Exercise 10.** Show that if $E \subseteq \mathbb{R}^d$, then there exists a $G_\delta$-set $H \supseteq E$ such that

$$|E|_e \ =\ |H|_e.$$

Thus, every subset $E$ of $\mathbb{R}^d$ is contained in a $G_\delta$-set $H$ that has exactly the same exterior Lebesgue measure as $E$.

### 1.4.3 Outer Measures

Now let us use the example of exterior Lebesgue measure to see how to define more general outer measures.

**Definition 11** (Outer Measure)**.** Let $X$ be a nonempty set. An *outer measure* or *exterior measure* on $X$ is a function $\mu^* \colon \mathcal{P}(X) \to [0,\infty]$ that satisfies the following conditions.

(a) $\mu^*(\emptyset) = 0$.

(b) Monotonicity: If $A \subseteq B$, then $\mu^*(A) \le \mu^*(B)$.

(c) Countable subadditivity: If $A_1, A_2, \ldots \subseteq X$, then

$$\mu^*\Bigl(\bigcup_i A_i\Bigr) \ \le\ \sum_i \mu^*(A_i).$$

We want to show now that if we are given any particular class of "elementary sets" whose measures are specified, then we can extend this to an outer measure on $X$.

**Theorem 12.** Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be any fixed collection of sets such that $\emptyset \in \mathcal{E}$ and there exist countably many $E_k \in \mathcal{E}$ such that $\cup E_k = X$ (we refer to the elements of $\mathcal{E}$ as *elementary sets*). Suppose that $\rho\colon \mathcal{E} \to [0,\infty]$ satisfies $\rho(\emptyset) = 0$. For each $A \subseteq X$, define

$$\mu^*(A) \;=\; \inf\Bigl\{\sum_k \rho(E_k)\Bigr\}, \tag{2}$$

where the infimum is taken over all finite or countable covers of $A$ by sets $E_k \in \mathcal{E}$. Then $\mu^*$ is an outer measure on $X$.

*Proof.* The given hypotheses ensure that every subset of $X$ has at least one covering by elements of $\mathcal{E}$. Hence the infimum in equation (2) is not taken over the empty set, and therefore does define a value in $[0,\infty]$ for each $A \subseteq X$.

Since $\{\emptyset\}$ is one covering of $\emptyset$ by elements of $\mathcal{E}$, we have that

$$0 \;\le\; \mu^*(\emptyset) \;\le\; \rho(\emptyset) \;=\; 0.$$

To show monotonicity, suppose that $A \subseteq B \subseteq X$. Since every covering of $B$ by sets $E_k \in \mathcal{E}$ is also a covering of $A$, it follows immediately from the definition of $\mu^*$ that $\mu^*(A) \le \mu^*(B)$.

To prove countable subadditivity, suppose that $A_1, A_2, \ldots \subseteq X$ are given, and fix any $\varepsilon > 0$. Then there exist set $E_j^{(k)} \in \mathcal{E}$ such that $A_k \subseteq \cup_j E_j^{(k)}$ and

$$\sum_j \rho(E_j^{(k)}) \;\le\; \mu^*(A_k) + \frac{\varepsilon}{2^k}.$$

Then

$$\bigcup_k A_k \;\subseteq\; \bigcup_{k,j} E_j^{(k)},$$

so

$$\mu^*\Bigl(\bigcup_k A_k\Bigr) \;\le\; \sum_{k,j} \rho(E_j^{(k)}) \;\le\; \sum_k \mu^*(A_k) \;+\; \varepsilon.$$

Since $\varepsilon$ is arbitrary, countable subadditivity follows, and therefore we have shown that $\mu^*$ is an outer measure on $X$. □

Despite the fact that we have shown that $\mu^*$ is an outer measure on $X$, there is still an important point to make: We have *not* shown that $\mu^*(E) = \rho(E)$ for $E \in \mathcal{E}$. The analogous fact for exterior Lebesgue measure is Theorem 7, which showed that if $Q$ is a cube in $\mathbb{R}^d$, then $|Q|_e = \mathrm{vol}(Q)$. However, note that to prove that equality, we used some topological facts regarding cubes. Thus we made use of the fact that $\mathbb{R}^d$ is not only a set, but is a topological space, and furthermore that cubes can be used as "building blocks" to construct the open subsets of $\mathbb{R}^d$. Hence it is perhaps not surprising that in a completely general setting we cannot prove that $\mu^*(E) = \rho(E)$ for arbitrary outer measures. And in fact this is a point that we will have to return to — what extra conditions will we need in order to ensure that we will have $\mu^*(E) = \rho(E)$ for $E \in \mathcal{E}$? This is a critical point, for we are trying to find a measure that *extends* the values $\rho(E)$ imposed on the elementary sets. We don't want to construct a measure that has no relation to our elementary building blocks, we want it to be consistent with them; otherwise it will have limited use for us.

### 1.4.4 Lebesgue Measurable Sets

At this point, we have constructed an *exterior* Lebesgue measure on all subsets of $\mathbb{R}^d$. We now turn to the delicate task of finding an appropriate smaller $\sigma$-algebra $\mathcal{L}$ to restrict to in order to turn this exterior measure into a true measure. Later, we will try to find an analogous restriction for general outer measures $\mu^*$ on $X$.

Recall Theorem 9, which said that given any $E \subseteq \mathbb{R}^d$ and any $\varepsilon > 0$, we can find an open set $U \supseteq E$ such that

$$|U|_e \;\le\; |E|_e + \varepsilon. \tag{3}$$

Note that we can write

$$U \;=\; E \cup (U \backslash E),$$

and furthermore this is a *disjoint* union. By subadditivity, we have that

$$|U|_e \;=\; \bigl|E \cup (U\backslash E)\bigr|_e \;\le\; |E|_e + |U\backslash E|_e. \tag{4}$$

Yet, by themselves, equations (3) and (4) do *not* imply that

$$|U\backslash E|_e \;\le\; \varepsilon \qquad \leftarrow \text{WE DO NOT KNOW THIS!}$$

On the other hand, *if* we knew that

$$|U|_e \;=\; |E|_e + |U\backslash E|_e \qquad \leftarrow \text{WE DO NOT KNOW THIS EITHER!}$$

*then* we would have enough information to conclude that $|U\backslash E|_e \le \varepsilon$. However, we just don't have that information, and in fact we will see examples later where this is *false*. There do exist sets $E \subseteq \mathbb{R}^d$ such that for some $\varepsilon > 0$ we can find an open set $U \supseteq E$ with

$$|E|_e \;\le\; |U|_e \;\le\; |E|_e + \varepsilon \qquad \text{yet} \qquad |U\backslash E|_e > \varepsilon.$$

These are very strange sets indeed. Perhaps if we ignore them they won't bother us too much. Let's pretend they don't exist — or, more precisely, let us restrict our attention to those sets that do *not* have this strange behavior.

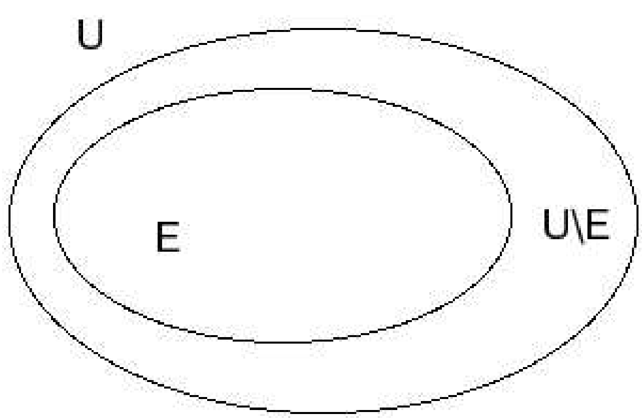


Figure 1. $E \subseteq U$.

**Definition 13.** A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable*, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U \backslash E|_e \le \varepsilon.$$

If $E$ is Lebesgue measurable, then its *Lebesgue measure* $|E|$ is its exterior measure, i.e., we set

$$|E| \;=\; |E|_e$$

for those sets $E$ that are Lebesgue measurable.

We define

$$\mathcal{L} \;=\; \mathcal{L}(\mathbb{R}^d) \;=\; \{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\},$$

and we refer to $\mathcal{L}$ as the *Lebesgue $\sigma$-algebra on* $\mathbb{R}^d$ (though we have not yet proved that it is a $\sigma$-algebra!).

We emphasize again that we can *always* find a open set $U \supseteq E$ with $|E|_e \le |U|_e \le |E|_e + \varepsilon$. However, only for measurable sets can we be sure that we have the further inequality $|U \backslash E|_e \le \varepsilon$.

Note that every open set $U$ is Lebesgue measurable, because $U \supseteq U$ and

$$|U \backslash U|_e \;=\; 0 \;<\; \varepsilon.$$

Here is a another example of a measurable set.

**Lemma 14.** If $Z \subseteq \mathbb{R}^d$ and $|Z|_e = 0$ then $Z$ is Lebesgue measurable.

*Proof.* Suppose that $|Z|_e = 0$. If we choose $\varepsilon > 0$, then we can find an open set $U \supseteq Z$ such that

$$|U|_e \;\le\; |Z|_e + \varepsilon \;=\; 0 + \varepsilon \;=\; \varepsilon.$$

Since $U \backslash Z \subseteq U$, we therefore have by monotonicity that

$$|U \backslash Z|_e \;\le\; |U|_e \;\le\; \varepsilon.$$

Hence $Z$ is measurable. $\square$

Thus, *every set with zero exterior measure is Lebesgue measurable.* This includes every countable set, so, for example, $\mathbb{Q}$ is a measurable subset of $\mathbb{R}$. On the other hand, uncountable sets may also have measure zero. For example, the Cantor set $C$ is an uncountable subset of $\mathbb{R}$ that has exterior measure zero, and so $C$ is a Lebesgue measurable subset of $\mathbb{R}$.

Our ultimate goal is to show that the Lebesgue $\sigma$-algebra $\mathcal{L}$ really is a $\sigma$-algebra on $\mathbb{R}^d$, and that Lebesgue measure really is a measure with respect to this $\sigma$-algebra. Note that since every subset of a set with exterior measure zero also has exterior measure zero and hence is measurable, Lebesgue measure will be *complete* in the sense introduced in Section 1.3.

To begin, let us prove that $\mathcal{L}$ is closed under countable unions.

**Theorem 15** (Closure Under Countable Unions)**.** If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$ are Lebesgue measurable, then so is $E = \cup E_k$, and

$$|E| \;\le\; \sum_{k=1}^{\infty} |E_k|.$$

*Proof.* Fix $\varepsilon > 0$. Since $E_k$ is measurable, there exists an open set $U_k \supseteq E_k$ such that

$$|U_k \backslash E_k|_e \;\le\; \frac{\varepsilon}{2^k}.$$

Then $U = \cup U_k$ is an open set, $U \supseteq E$, and

$$U \backslash E \;=\; \Big(\bigcup_k U_k\Big) \setminus \Big(\bigcup_k E_k\Big) \;\subseteq\; \bigcup_k (U_k \backslash E_k).$$

Hence

$$|U \backslash E|_e \;\le\; \sum_{k=1}^{\infty} |U_k \backslash E_k|_e \;\le\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon,$$

so $E$ is measurable. Since we already know that exterior measure is subadditive, we have that $|E| \;\le\; \sum |E_k|$. $\square$

To complete the proof that $\mathcal{L}$ is a $\sigma$-algebra, we need to show that it is closed under complements. However, it turns out to take a fair amount of work to do it directly. We will do it later (see Theorem 31), but at this point I ask your indulgence — let us simply accept on faith for the moment that $\mathcal{L}$ is closed under complements, and move on to see what further results follow from this.

**Axiom 16** (Closure Under Complements)**.** We take as an axiom that $\mathcal{L}$ is closed under complements. That is, if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then so is $E^{\mathrm{C}} = \mathbb{R}^d \backslash E$.

In particular, accepting Axiom 16, we have that $\mathcal{L}$ is a $\sigma$-algebra on $\mathbb{R}^d$. Since $\mathcal{L}$ contains all the open sets, it therefore also contains all of the closed sets. And since $\mathcal{L}$ is also closed under countable unions and intersections, it follows that $\mathcal{L}$ contains all the $G_\delta$ sets (finite or countable intersections of open sets) and all the $F_\sigma$ sets (finite or countable unions of closed sets). In fact, we have the following facts about the Lebesgue $\sigma$-algebra.

**Lemma 17.**

(a) If $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^d$, then $\mathcal{B} \subseteq \mathcal{L}$.

(b) If any $E \in \mathcal{L}$ satisfies $|E| = 0$, then every subset $A \subseteq E$ belongs to $\mathcal{L}$ and also satisfies $|A| = 0$.

**Remark 18.** It can be shown that the cardinality of $\mathcal{B}$ equals the cardinality of the real line $\mathbb{R}$. On the other hand, the Cantor set $C$ also satisfies $\mathrm{card}(C) = \mathrm{card}(\mathbb{R})$, and *every* subset of $C$ is measurable, so $\mathrm{card}(\mathcal{L}) \ge \mathrm{card}(\mathcal{P}(C)) > \mathrm{card}(\mathbb{R})$. Hence there exist Lebesgue measurable sets that are not Borel sets, so $\mathcal{B} \subsetneq \mathcal{L}$.

Assuming that $\mathcal{L}$ is indeed a $\sigma$-algebra, we can derive some equivalent formulations of what it means to be a Lebesgue measurable set. To motivate this, recall that a Exercise 10 showed that given an arbitrary set $E \subseteq \mathbb{R}^d$, we can find a $G_\delta$-set $H \supseteq E$ such that $|E|_e = |H|$. However, just as in our discussion leading up to the definition of Lebesgue measurable sets, we *cannot* conclude from this that $|H \backslash E|_e = 0$! And indeed, the next result shows that this is an equivalent way of distinguishing Lebesgue measurable sets from nonmeasurable sets.

**Theorem 19.** If $E \subseteq \mathbb{R}^d$ is given, then the following statements are equivalent.

(a) $E$ is Lebesgue measurable.

(b) For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E \backslash F|_e \leq \varepsilon$.

(c) $E = H \backslash Z$ where $H$ is a $G_\delta$-set and $|Z| = 0$.

(d) $E = H \cup Z$ where $H$ is an $F_\sigma$-set and $|Z| = 0$.

*Proof.* (a) $\iff$ (b). This follows from the fact that $\mathcal{L}$ is closed under complements. Exercise: Fill in the details.

(a) $\Rightarrow$ (c). Suppose that $E$ is measurable. Then for each $k \in \mathbb{N}$ we can find an open set $U_k \supseteq E$ such that $|U_k \backslash E| < 1/k$. Let $H = \cap U_k$. Then $H$ is a $G_\sigma$-set, $H \supseteq E$, and $Z = H \backslash E \subseteq U_k \backslash E$ for every $k$. Hence $|Z|_e \leq |U_k \backslash E| < 1/k$ for every $k$, so $|Z| = 0$.

Exercise: Complete the remaining implications. □

We will not prove it, but the following is a very useful standard fact about open sets. We say that cubes are *nonoverlapping* if their interiors are disjoint.

**Theorem 20.** In $\mathbb{R}^d$, every open set can be written as a countable union of nonoverlapping cubes. That is, if $U \subseteq \mathbb{R}^d$ is open, then there exist cubes $\{Q_k\}_k$ with disjoint interiors such that $U = \cup Q_k$.

Using the preceding theorem we can derive another characterization of measurability.

**Exercise 21.** Let $E \subseteq \mathbb{R}^d$ satisfy $|E|_e < \infty$. Show that $E$ is measurable if and only if for each $\varepsilon > 0$ we can write

$$E = (S \cup N_1) \setminus N_2,$$

where $S$ is a union of finitely many nonoverlapping cubes, and $|N_1|_e$, $|N_2|_e < \varepsilon$.

### 1.4.5 Lebesgue Measure

Now we will prove that Lebesgue measure is a measure on $\mathbb{R}^d$ with respect to the Lebesgue $\sigma$-algebra $\mathcal{L}$. In the process, we will also fill in the proof that $\mathcal{L}$ is closed under complements. Therefore, at this point we retract our acceptance of Axiom 16 and proceed to prove it and other facts about Lebesgue measure.

One basic fact that we will need is that cubes are measurable.

**Exercise 22.** Let $Q$ be a cube in $\mathbb{R}^d$. Prove that $|\partial Q|_e = 0$, and use this to show that $Q$ is measurable and that $|Q| = |Q^\circ|$.

We already know that Lebesgue measure is countably subadditive, so our ultimate goal is to show that it is countably additive. Let us begin with the special case of finite unions of cubes. Again, the meaning of "nonoverlapping cubes" is that they have disjoint interiors, and recall that we proved in Theorem 7 that $|Q| = \mathrm{vol}(Q)$ for any cube $Q$. In order to prove that the measure of a finite union of nonoverlapping cubes is the sum of the measures of the cubes, we need the following exercise.

**Exercise 23.** Let $Q_1$, $Q_2$, and $R$ be arbitrary cubes in $\mathbb{R}^d$ (see Figure 2). Show that

$$\mathrm{vol}(R) \;\geq\; \mathrm{vol}(R \cap Q_1) + \mathrm{vol}(R \cap Q_2) - \mathrm{vol}(R \cap Q_1 \cap Q_2).$$

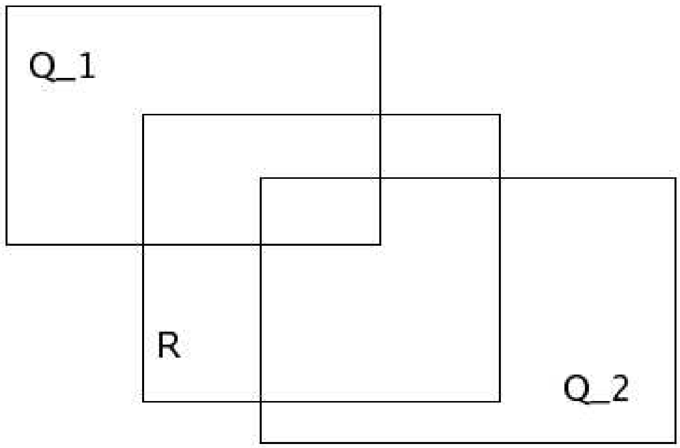


FIGURE 2. Cubes $Q_1$, $Q_2$, and $R$ for Exercise 23.

**Lemma 24.** If $\{Q_k\}_{k=1}^N$ is a finite collection of nonoverlapping cubes, then

$$\left|\bigcup_{k=1}^N Q_k\right| \;=\; \sum_{k=1}^N |Q_k|.$$

*Proof.* We know that the finite union $Q = \cup_{k=1}^N Q_k$ is measurable since each cube is measurable. Further, by subadditivity we have

$$|Q| \;=\; \left|\bigcup_{k=1}^N Q_k\right| \;\leq\; \sum_{k=1}^N |Q_k|.$$

For simplicity of presentation, we will establish the opposite inequality for the case $N = 2$ only. Let $Q_1$, $Q_2$ be nonoverlapping cubes, and suppose that $\{R_\ell\}_\ell$ is any cover of $Q_1 \cup Q_2$ by countably many cubes. Note that $\{R_\ell \cap Q_1\}_\ell$ is then a covering of $Q_1$ by cubes, so we have

$$|Q_1| \;\leq\; \sum_\ell |R_\ell \cap Q_1|,$$

and similarly for $Q_2$. Also, since $Q_1$ and $Q_2$ are nonoverlapping, we have that

$$|Q_1 \cap Q_2| \;=\; 0.$$

Therefore, Exercise 23 implies that

$$|R_\ell| \;\geq\; |R_\ell \cap Q_1| + |R_\ell \cap Q_2|,$$

and therefore

$$|Q_1| + |Q_2| \;\le\; \sum_\ell |R_\ell \cap Q_1| + \sum_\ell |R_\ell \cap Q_2| \;\le\; \sum_\ell |R_\ell|.$$

Since this is true for every covering of $Q_1 \cup Q_2$, we conclude that

$$|Q_1| + |Q_2| \;\le\; \inf\Bigl\{\sum_\ell |R_\ell|\Bigr\} \;=\; |Q_1 \cup Q_2|,$$

where the infimum is taken over all the possible coverings of $Q_1 \cup Q_2$ by countably many cubes.

Exercise: Extend the proof to arbitrary $N$. □

**Definition 25.** The *distance* between two sets $A$, $B \subseteq \mathbb{R}^d$ is

$$d(A,B) \;=\; \inf\bigl\{|x-y| : x \in A, y \in B\bigr\}.$$

We will show next that additivity holds for any two sets that are separated by a positive distance. In fact, this is even true for *exterior* Lebesgue measure, i.e., it is true regardless of whether the two sets are measurable or not.

**Lemma 26.** If $A$, $B \subseteq \mathbb{R}^d$ satisfy $d(A,B) > 0$, then

$$|A \cup B|_e \;=\; |A|_e + |B|_e.$$

*Proof.* By subadditivity, we have

$$|A \cup B|_e \;\le\; |A|_e + |B|_e.$$

To show the opposite inequality, fix any $\varepsilon > 0$. Then by definition of exterior measure, there exist cubes $Q_k$ such that $A \cup B \subseteq \cup Q_k$ and

$$\sum_k |Q_k| \;\le\; |A \cup B|_e + \varepsilon.$$

By dividing each $Q_k$ into subcubes if necessary, we can assume that the diameter of each $Q_k$ is less than $d(A,B)$, i.e.,

$$\operatorname{diam}(Q_k) \;=\; \sup\bigl\{|x-y| : x, y \in Q_k\bigr\} \;<\; d(A,B).$$

Consequently, each $Q_k$ can intersect at most one of $A$ or $B$.

Let $\{Q_k^A\}_k$ be the subsequence of $\{Q_k\}_k$ that contains those cubes that intersect $A$, and $\{Q_k^B\}_k$ the subsequence of cubes that intersect $B$. Since $\{Q_k\}_k$ covers $A \cup B$, we must have

$$A \;\subseteq\; \bigcup_k Q_k^A \qquad \text{and} \qquad B \;\subseteq\; \bigcup_k Q_k^B.$$

Therefore

$$|A|_e + |B|_e \;\le\; \sum_k |Q_k^A| \;+\; \sum_k |Q_k^B| \;\le\; \sum_k |Q_k| \;\le\; |A \cup B|_e.$$

Since $\varepsilon$ is arbitrary, we conclude that $|A|_e + |B|_e \le |A \cup B|_e$. □

**Exercise 27.** Prove the undergraduate real analysis fact that

$$A, B \text{ disjoint and compact} \quad \Longrightarrow \quad d(A,B) > 0.$$

The following corollary then follows by induction.

**Corollary 28.** *If $F_1, \dots, F_N$ are finitely many disjoint compact sets, then*

$$\Big|\bigcup_{k=1}^N F_k\Big|_e \;=\; \sum_{k=1}^N |F_k|_e.$$

If we accept Axiom 16, then we know that all compact sets are measurable since they are complements of open sets. However, since we are no longer accepting Axiom 16, we do not yet know whether compact sets are measurable. Hence, in the preceding corollary we have written exterior measure instead of Lebesgue measure. Let us now work on rectifying our omission of the proof of Axiom 16.

**Theorem 29.** *Every compact set $F \subseteq \mathbb{R}^d$ is Lebesgue measurable.*

*Proof.* Since $F$ is closed and bounded, we can find a single cube $Q$ large enough that $F \subseteq Q$. Hence $|F|_e \le |Q| < \infty$.

Now fix any $\varepsilon > 0$. Then there exists an open set $U \supseteq F$ such that

$$|F|_e \;\le\; |U| + |F|_e + \varepsilon.$$

We must show that $|U \backslash F|_e \le \varepsilon$.

Since $U\backslash F$ is open, there exist countably many nonoverlapping cubes $Q_k$ such that

$$U \backslash F \;=\; \bigcup_k Q_k.$$

For any finite $N$, the set

$$R_N \;=\; \bigcup_{k=1}^N Q_k$$

is compact since each $Q_k$ is compact. Further, by Lemma 24,

$$|R_N| \;=\; \sum_{k=1}^N |Q_k|.$$

Also, $R_N \subseteq U\backslash F$, so $R_N$ and $F$ are disjoint compact sets. Therefore, by Corollary 28,

$$|F|_e + \sum_{k=1}^N |Q_k| \;=\; |F|_e + |R_N| \;=\; |F \cup R_N|_e \;\le\; |F \cup (U\backslash F)|_e \;=\; |U|.$$

This is true for every $N$. Since all of the quantities appearing above are finite, we can rearrange and combine this with subadditivity to compute that

$$|U\backslash F|_e \;\le\; \sum_k |Q_k| \;=\; \lim_{N\to\infty} \sum_{k=1}^N |Q_k| \;\le\; |U| - |F|_e \;\le\; \varepsilon. \qquad \square$$

**Corollary 30.** Every closed set $F \subseteq \mathbb{R}^d$ is Lebesgue measurable.

*Proof.* Let $B_k = \{x \in \mathbb{R}^d : |x| \le k\}$ be the closed ball in $\mathbb{R}^d$ of radius $k$ centered at the origin. Then $F_k = F \cap B_k$ is compact and hence is measurable. Since $F$ is the union of the countably many sets $F_k$, we conclude that $F$ is measurable as well. □

Finally, we can give the neglected proof of Axiom 16.

**Theorem 31** (Closure Under Complements)**.** If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then so is $E^{\mathrm{C}} = \mathbb{R}^d \backslash E$.

*Proof.* Suppose that $E$ is measurable. Then for each $k$ we can find an open $U_k \supseteq E$ such that $|U_k \backslash E|_e < 1/k$. Define

$$F_k \;=\; U_k^{\mathrm{C}}.$$

Then $F_k$ is closed and hence is measurable. Let

$$H \;=\; \bigcup_k F_k \;=\; \bigcup_k U_k^{\mathrm{C}}.$$

Then $H$ is measurable and $H \subseteq E^{\mathrm{C}}$. Let $Z = E^{\mathrm{C}} \backslash H$. Then for any fixed $j$ we have

$$Z \;=\; E^{\mathrm{C}} \setminus \bigcup_k U_k^{\mathrm{C}} \;\subseteq\; E^{\mathrm{C}} \backslash U_j^{\mathrm{C}} \;=\; U_j \backslash E.$$

Hence for every $j$ we have

$$|Z|_e \;\le\; |U_j \backslash E|_e \;<\; \frac{1}{j},$$

and therefore $|Z|_e = 0$. But then $Z$ is measurable, so $E^{\mathrm{C}} = H \cup Z$ is measurable as well. □

As a consequence, we know now that all of the earlier results that relied on on Axiom 16 are valid (Lemma 17, Theorem 19, and Exercise 21).

Finally, we show that Lebesgue measure is indeed a measure.

**Theorem 32** (Countable Additivity of Lebesgue Measure)**.** If $E_1, E_2, \dots$ are disjoint Lebesgue measurable subsets of $\mathbb{R}^d$, then

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

*Proof.* First let us assume that each $E_k$ is a bounded set. By subadditivity, we have

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \;\le\; \sum_{k=1}^{\infty} |E_k|,$$

so our goal is to prove the opposite inequality.

Fix any $\varepsilon > 0$. Then since $E_k$ is measurable, by Theorem 19 we can find a closed set $F_k \subseteq E_k$ such that

$$|E_k \backslash F_k| \;<\; \frac{\varepsilon}{2^k}.$$

Note that the $F_k$ are disjoint compact sets. Therefore, for any finite $N$ we have

$$\sum_{k=1}^{N} |F_k| \;=\; \Bigl|\bigcup_{k=1}^{N} F_k\Bigr| \;\le\; \Bigl|\bigcup_{k=1}^{N} E_k\Bigr| \;\le\; \Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr|,$$

where the last two inequalities follow from monotonicity. Since this is true for every $N$, we conclude that

$$\sum_{k=1}^{\infty} |F_k| \;\le\; \Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr|.$$

Finally,

$$\begin{aligned}
\sum_{k=1}^{\infty} |E_k| \;=\; \sum_{k=1}^{\infty} |F_k \cup (E_k \backslash F_k)| \;&\le\; \sum_{k=1}^{\infty} \bigl(|F_k| + |E_k \backslash F_k|\bigr) \\
&\le\; \sum_{k=1}^{\infty} \Bigl(|F_k| + \frac{\varepsilon}{2^k}\Bigr) \\
&=\; \sum_{k=1}^{\infty} |F_k| \;+\; \varepsilon \\
&\le\; \Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr| \;+\; \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, this shows that

$$\sum_{k=1}^{\infty} |E_k| \;\le\; \Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr|.$$

This completes the proof for the case where each $E_k$ is bounded.

Now suppose that the $E_k$ are arbitrary disjoint measurable sets. For each $j,\, k \in \mathbb{N}$, set

$$E_k^j \;=\; \{x \in E_k : j-1 \le |x| < j\}.$$

Then $\{E_k^j\}_{k,j}$ is a countable collection of disjoint measurable sets, and for each $k$ we have $\cup_j E_k^j = E_k$. Since each $E_k^j$ is bounded, by applying the equality for bounded sets twice we obtain

$$\Bigl|\bigcup_{k} E_k\Bigr| \;=\; \Bigl|\bigcup_{k}\bigcup_{j} E_k^j\Bigr| \;=\; \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |E_k^j| \;=\; \sum_{k=1}^{\infty} |E_k|. \qquad \square$$

**Corollary 33.** $\mathcal{L}$ is a $\sigma$-algebra on $\mathbb{R}^d$, and $\mu(E) = |E|$ for $E \in \mathcal{L}$ defines a measure on $\mathbb{R}^d$. Furthermore, $(\mathbb{R}^d, |\cdot|, \mathcal{L})$ is complete, and $\mathcal{L}$ contains the Borel $\sigma$-algebra $\mathcal{B}$ on $\mathbb{R}^d$.

### 1.4.6 Carathéodory's Criterion for Lebesgue Measurability

Though we have finished constructing Lebesgue measure on $\mathbb{R}^d$, there is still an important observation to make that will motivate the development of abstract measure theory.

Take another look at Theorem 19, which gives several equivalent characterizations of Lebesgue measurability. Each of these characterizations are topologically-related in some way, as they are formulated in terms of open, closed, $G_\delta$, or $F_\sigma$ sets. In contrast, the statement of the next characterization involves only the definition of outer measure. This makes this seemingly esoteric characterization quite important.

**Theorem 34** (Carathéodory's Criterion)**.** Let $A \subseteq \mathbb{R}^d$ be given. Then $A$ is measurable if and only if

$$\forall\, E \subseteq \mathbb{R}^d, \quad |E|_e \;=\; |E \cap A|_e + |E \backslash A|_e. \tag{5}$$

Before proving the theorem, let us note that a measurable set $A$ must have the property that when *any* other set $E$ is given, the exterior measures of the two disjoint pieces $E \cap A$ and $E\backslash A$ that $A$ cuts $E$ into must exactly add up to the exterior measure of $E$. This has to be true for every set $E$, measurable or not.

*Proof of Theorem 34.*

$\Rightarrow$. Suppose that $A$ is measurable and that $E$ is any subset of $\mathbb{R}^d$. Note that since $E = (E \cap A) \cup (E\backslash A)$, we have by subadditivity that

$$|E|_e \;\le\; |E \cap A|_e + |E\backslash A|_e.$$

So, we just have to establish the opposite inequality.

By Exercise 10, we can find a $G_\delta$-set $H \supseteq E$ such that $|H| = |E|_e$. Note that we can write $H$ as the disjoint union

$$H \;=\; (H \cap A) \cup (H\backslash A).$$

Since Lebesgue measure is countably additive and $H$, $A$ are measurable, we therefore have that

$$\begin{aligned} |E|_e \;=\; |H| \;&=\; |H \cap A| + |H\backslash A| \qquad \text{(additivity)} \\ &\ge\; |E \cap A| + |E\backslash A| \qquad \text{(monotonicity)}. \end{aligned}$$

$\Leftarrow$. Suppose that the Carathéodory criterion (5) holds.

Let us assume first that $A$ is bounded. Let $H \supseteq A$ be a $G_\delta$-set such that $|H| = |A|_e$. Then by equation (5),

$$|A|_e \;=\; |H| \;=\; |H \cap A|_e + |H\backslash A|_e \;=\; |A|_e + |H\backslash A|_e.$$

Since $|A|_e < \infty$, we conclude that $Z = H\backslash A$ has zero exterior measure and hence is measurable. Since $A = H\backslash Z$, it is measurable as well.

Now let $A$ be arbitrary. For each $k$, let

$$A_k \;=\; \{x \in A : |x| \le k\}.$$

For each $k$, there exists a $G_\delta$-set $H_k \supseteq A_k$ such that $|H_k| = |A_k|_e$. Applying equation (5), we conclude that

$$|A_k|_e \;=\; |H_k| \;=\; |H_k \cap A|_e + |H_k \backslash A|_e \;\geq\; |A_k|_e + |H_k \backslash A|_e.$$

Since $|A_k|_e < \infty$, we conclude that $Z_k = H_k \backslash A$ has exterior measure zero and hence is measurable. Let

$$H \;=\; \bigcup_k H_k.$$

Then $H$ is measurable, and

$$Z \;=\; H \backslash A \;=\; \Bigl(\bigcup_k H_k\Bigr) \setminus A \;\subseteq\; \bigcup_k (H_k \backslash A) \;=\; \bigcup_k Z_k.$$

Hence $|Z|_e \leq \sum |Z_k| = 0$, so $Z$ is measurable. But then $A = H \backslash Z$ is measurable as well. $\square$

### 1.4.7 Outer Measures Revisited

Now that we have fully developed Lebesgue measure, let us return to consideration of abstract measures. Given an arbitrary outer measure $\mu^*$ on a set $X$, our goal is to create a $\sigma$-algebra $\mathcal{M}$ on $X$ such that $\mu^*$ *restricted to* $\mathcal{M}$ will be countably additive. The elements of $\mathcal{M}$ are our "good sets," the sets that are measurable with respect to $\mu^*$. But how do we define measurability for an arbitrary outer measure? There need not be any topology on $X$, so we do not have a natural analogue of the definition of Lebesgue measurable sets (Definition 13). On the other hand, the equivalent formulation of Lebesgue measurability given by Carathéodory's Criterion (Theorem 34) does not involve topology, and as such it is the appropriate motivation for the following definition.

**Definition 35** (Measurable Set)**.** Let $\mu^*$ be an outer measure on a set $X$. Then a set $A \subseteq X$ is *$\mu^*$-measurable*, or simply *measurable* for short, if

$$\forall\, E \subseteq X, \quad \mu^*(E) \;=\; \mu^*(E \cap A) + \mu^*(E \backslash A).$$

It is often convenient to use the equality

$$\mu^*(E \cap A) + \mu^*(E \backslash A) \;=\; \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}).$$

Observe that the empty set is $\mu^*$-measurable by virtue of the fact that $\mu^*(\emptyset) = 0$.

Note that, by subadditivity, we always have the inequality

$$\mu^*(E) \;\leq\; \mu^*(E \cap A) + \mu^*(E \backslash A).$$

Hence, to establish measurability, we just have to prove that the opposite inequality holds for every $E \subseteq X$.

As is true for exterior Lebesgue measure, every subset with outer measure zero is measurable.

**Lemma 36.** Let $\mu^*$ be an outer measure on $X$. Then every set $A \subseteq X$ with $\mu^*(A) = 0$ is $\mu^*$-measurable.

*Proof.* Suppose that $\mu^*(A) = 0$. If $E$ is any subset of $X$, then

$$\begin{aligned} \mu^*(E) &\le \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}) && \text{(subadditivity)} \\ &= 0 + \mu^*(E \cap A^{\mathrm{C}}) && \text{(monotonicity)} \\ &\le \mu^*(E) && \text{(monotonicity).} \end{aligned}$$

Hence $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}})$, and so $A \in \mathcal{M}$. □

Now we prove that the measurable sets form a $\sigma$-algebra, and that $\mu^*$ restricted to this $\sigma$-algebra forms a complete measure. Note that Lebesgue measure is just a special case of this theorem, so we are certainly duplicating some effort here. However, the additional insights gained from considering Lebesgue measure first make this duplication worthwhile.

**Theorem 37** (Carathéodory's Theorem)**.** If $\mu^*$ is an outer measure on a set $X$, then

$$\mathcal{M} \;=\; \bigl\{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\bigr\}$$

is a $\sigma$-algebra, and $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure (in fact, even more is true: *every* set $E \subseteq X$ with $\mu^*(E) = 0$ is $\mu^*$-measurable).

*Proof.* a. Since the empty set is $\mu^*$-measurable, we know that $\mathcal{M}$ is not empty.

b. To show that $\mathcal{M}$ is closed under complements, fix any $A \in \mathcal{M}$, and let $E \subseteq X$ be arbitrary. Then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \backslash A) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}) \\ &= \mu^*(E \cap A^{\mathrm{C}}) + \mu^*(E \cap (A^{\mathrm{C}})^{\mathrm{C}}) \\ &= \mu^*(E \cap A^{\mathrm{C}}) + \mu^*(E \backslash A^{\mathrm{C}}). \end{aligned}$$

Hence $A^{\mathrm{C}}$ is measurable, so $A^{\mathrm{C}} \in \mathcal{M}$.

c. Ultimately we want to show that $\mathcal{M}$ is closed under countable unions, but to begin with let us show that it is closed under finite unions. By induction, it suffices to show that if $A,\, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

Choose any set $E \subseteq X$. By subadditivity, we have

$$\mu^*(E) \;\le\; \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^{\mathrm{C}}).$$

Applying the fact that $A,\, B \in \mathcal{M}$ and the subadditivity of $\mu^*$, we have:

$$\begin{aligned} &\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^{\mathrm{C}}). \\ &\quad= \mu^*\bigl((E \cap A \cap B) \,\cup\, (E \cap A \cap B^{\mathrm{C}}) \,\cup\, (E \cap A^{\mathrm{C}} \cap B)\bigr) + \mu^*(E \cap (A \cup B)). \\ &\quad\le \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^{\mathrm{C}}) + + \mu^*(E \cap A^{\mathrm{C}} \cap B) + \mu^*(E \cap A^{\mathrm{C}} \cap B^{\mathrm{C}}) \\ &\quad= \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}) \\ &\quad= \mu^*(E) \end{aligned}$$

Therefore $A \cup B \in \mathcal{M}$.

d. Suppose that $A$, $B \in \mathcal{M}$ are *disjoint*. The previous part showed that $A \cup B$ is $\mu^*$-measurable, so

$$\mu^*(A \cup B) \;=\; \mu^*\big((A \cup B) \cap A\big) + \mu^*\big((A \cup B) \cap A^{\mathrm{C}}\big) \;=\; \mu^*(A) + \mu^*(B).$$

Thus, $\mu^*$ is finitely additive *on* $\mathcal{M}$. However, we do not know whether $\mu^*$ is finitely additive for all subsets of $X$.

e. Now we will show that $\mathcal{M}$ is closed under countable unions. By an observation from Section 1.2, it suffices to assume that $A_1, A_2, \dots \in \mathcal{M}$ are *disjoint* sets, and to show that their union belongs to $\mathcal{M}$. Define

$$B \;=\; \bigcup_{k=1}^{\infty} A_k \qquad \text{and} \qquad B_n \;=\; \bigcup_{k=1}^{n} A_k, \quad n \in \mathbb{N}.$$

Note that $B_n \in \mathcal{M}$ since we have shown that $\mathcal{M}$ is closed under finite unions.

Choose any $E \subseteq X$. We claim that

$$\mu^*(E \cap B_n) \;=\; \sum_{k=1}^{n} \mu^*(E \cap A_k).$$

Note that this would be trivial if we knew that $\mu^*$ is finitely additive. However, we only know that $\mu^*$ is finitely additive on the $\mu^*$-measurable sets, so since $E$ is arbitrary we cannot use this fact.

Instead, we proceed by induction. Since $B_1 = A_1$, the claim is trivial for $n = 1$. Therefore, suppose that the claim holds for some $n \geq 1$. Then since $A_{n+1}$ is $\mu^*$-measurable,

$$\begin{aligned}
\mu^*(E \cap B_{n+1}) \;&=\; \mu^*\Big(E \,\cap\, \bigcup_{k=1}^{n+1} A_k\Big) \\
&=\; \mu^*\Big(E \,\cap\, \bigcup_{k=1}^{n+1} A_k \,\cap\, A_{n+1}\Big) + \mu^*\Big(E \,\cap\, \bigcup_{k=1}^{n+1} A_k \,\cap\, A_{n+1}^{\mathrm{C}}\Big) \\
&=\; \mu^*(E \cap A_{n+1}) + \mu^*(E \,\cap\, \bigcup_{k=1}^{n} A_n) \qquad \text{(by disjointness)} \\
&=\; \mu^*(E \cap A_{n+1}) + \sum_{k=1}^{n} \mu^*(E \cap A_k).
\end{aligned}$$

Hence the claim follows by induction.

Next,

$$\begin{aligned}
\sum_{k=1}^{n} \mu^*(E \cap A_k) \;+\; \mu^*(E \cap B^{\mathrm{C}}) \;&\leq\; \sum_{k=1}^{n} \mu^*(E \cap A_k) \;+\; \mu^*(E \cap B_n^{\mathrm{C}}) \qquad \text{(since } B^{\mathrm{C}} \subseteq B_n^{\mathrm{C}}) \\
&=\; \mu^*(E \cap B_n) + \mu^*(E \cap B_n^{\mathrm{C}}) \qquad \text{(by the claim)} \\
&=\; \mu^*(E) \qquad \text{(since } B_n \in \mathcal{M}).
\end{aligned}$$

As this is true for every $n$, we conclude that

$$\begin{aligned}
\mu^*(E) &\le \mu^*(E\cap B) + \mu^*(E\cap B^{\mathrm{C}}) && \text{(subadditivity)}\\
&\le \sum_{k=1}^{\infty} \mu^*(E\cap A_k) \;+\; \mu^*(E\cap B^{\mathrm{C}}) && \text{(subadditivity)}\\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*(E\cap A_k) \;+\; \mu^*(E\cap B^{\mathrm{C}})\\
&\le \mu^*(E) && \text{(from above).}
\end{aligned}$$

Thus we have that

$$\mu^*(E) \;=\; \sum_{k=1}^{\infty} \mu^*(E\cap A_k) \;+\; \mu^*(E\cap B^{\mathrm{C}}) \;=\; \mu^*(E\cap B) + \mu^*(E\cap B^{\mathrm{C}}), \tag{6}$$

so $B \in \mathcal{M}$.

f. To show that $\mu^*$ *restricted to* $\mathcal{M}$ is countably additive, let $A_1, A_2, \dots$ be disjoint sets in $\mathcal{M}$, and let $B = \cup A_k$. Then since we showed in the last part that $B$ is $\mu^*$-measurable, by setting $E = B$ in equation (6), we have that

$$\mu^*\Big(\bigcup_{k=1}^{\infty} A_k\Big) \;=\; \mu^*(B) \;=\; \sum_{k=1}^{\infty} \mu^*(B\cap A_k) \;+\; \mu^*(B\cap B^{\mathrm{C}}) \;=\; \sum_{k=1}^{\infty} \mu^*(A_k).$$

Hence $\mu^*$ is countably additive on $\mathcal{M}$, and therefore $\mu = \mu^*|_{\mathcal{M}}$ is a measure.

g. Finally, the fact that $\mu$ is a complete measure follows immediately from Lemma 36, since every subset of $X$ with zero outer measure is $\mu^*$-measurable. □

### 1.4.8 Premeasures

In the last section, we saw that given any outer measure $\mu^*$, there is an associated $\sigma$-algebra $\mathcal{M}$ of $\mu^*$-measurable sets, and that $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure. In Section 1.4.3 we discussed a method of constructing outer measures: Begin with any collection $\mathcal{E}$ of subsets of $X$ and any specified values $\rho(E)$ for $E \in \mathcal{E}$, and create the outer measure $\mu^*$ by setting

$$\mu^*(A) \;=\; \inf\Big\{\sum_k \rho(E_k)\Big\},$$

where the infimum is taken over all finite or countable covers of $A$ by sets $E_k \in \mathcal{E}$. This is exactly what we did to create Lebesgue measure. There we started with the collection of cubes in $\mathbb{R}^d$ and specified that $\rho(Q) = \mathrm{vol}(Q)$ for every cube $Q$. Then the exterior Lebesgue measure $\mu^*(A) = |A|_e$ of an arbitrary set $A \subseteq \mathbb{R}^d$ was defined by covering $A$ with countable collections of cubes.

However, for exterior Lebesgue we had some extra tools to work with, and using those we were able to show that all cubes were measurable and that $|Q|_e = \mathrm{vol}(Q) = \rho(Q)$ for every cube $Q$. We can ask whether analogous facts will hold for arbitrary outer measures created via this process. That is, suppose that $\mathcal{E}$ and $\rho$ are given and $\mu^*$ is as defined above, and consider the following questions.

- Will the elementary sets be measurable, i.e., will we have $\mathcal{E} \subseteq \mathcal{M}$?
- Will we have $\mu^*(E) = \rho(E)$ for $E \in \mathcal{E}$?

Unfortunately, the following example shows that the answers to these questions are no in general.

**Exercise 38.** Let $X$ be any set with at least two elements, and let $A$ be a nonempty proper subset of $A$. Set

$$\mathcal{E} \;=\; \{\emptyset, A, A^{\mathrm{C}}, X\}.$$

(a) Show that if we define

$$\rho(\emptyset) = 0, \qquad \rho(A) = \frac{1}{4}, \qquad \rho(A^{\mathrm{C}}) = \frac{1}{4}, \qquad \rho(X) = 1,$$

then

$$\mu^*(X) \;=\; \frac{1}{2} \;\neq\; \rho(X).$$

(b) Show that if we define

$$\rho(\emptyset) = 0, \qquad \rho(A) = 1 \qquad \rho(A^{\mathrm{C}}) = 1 \qquad \rho(X) = 1,$$

then

$$\mu^*(X) \;=\; 1 \;\neq\; 2 \;=\; \mu^*(X \cap A) + \mu^*(X \cap A^{\mathrm{C}}),$$

so $X$ is not $\mu^*$-measurable.

Thus, we need to impose some extra conditions on the function $\rho$ and the class $\mathcal{E}$ of elementary sets.

**Definition 39** (Premeasure)**.** Given a set $X$, let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, i.e., $\mathcal{A}$ is nonempty and is closed under complements and finite unions. Then a *premeasure* is a function $\mu_0 \colon \mathcal{A} \to [0, \infty]$ such that

(a) $\mu_0(\emptyset) = 0$, and

(b) if $A_1, A_2, \dots \in \mathcal{A}$ are disjoint and if $\cup A_k \in \mathcal{A}$, then

$$\mu_0\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr) \;=\; \sum_{k=1}^{\infty} \mu_0(A_k).$$

Note that in requirement (b) we are not assuming that $\mathcal{A}$ is closed under countable unions. We only require that *if* the union of the disjoint sets $A_k$ belongs to $\mathcal{A}$ *then* $\mu_0$ will be countably additive on those sets.

**Remark 40.** Note that $\mathcal{E} = \{Q : Q \text{ is a cube in } \mathbb{R}^d\}$ does not form an algebra, as it is not closed under either complements or finite unions. Thus it is not *entirely* obvious how the construction of Lebesgue measure relates to premeasures. We will consider this in Section 1.5.

**Lemma 41.** A premeasure $\mu_0$ is monotonic and finitely additive on $\mathcal{A}$.

*Proof.* To show finite additivity, suppose that $A_1, \dots, A_N \in \mathcal{A}$ are disjoint sets. Define $A_k = \emptyset$ for $k > 0$. Then $\cup A_k = \cup_{k=1}^N A_k \in \mathcal{A}$, so by definition of premeasure we have

$$\mu_0\Big(\bigcup_{k=1}^N A_k\Big) \;=\; \mu_0\Big(\bigcup_{k=1}^\infty A_k\Big) \;=\; \sum_{k=1}^\infty \mu_0(A_k) \;=\; \sum_{k=1}^N \mu_0(A_k).$$

Hence $\mu^*$ is finitely additive.

Next, if we have $A, B \in \mathcal{A}$ with $B \subseteq A$, then by finite additivity,

$$\mu_0(A) \;=\; \mu_0(B) + \mu_0(A \backslash B) \;\geq\; \mu_0(B).$$

Hence $\mu_0$ is monotonic on $\mathcal{A}$. □

Given any premeasure $\mu_0$, we associate the outer measure defined by

$$\mu^*(A) \;=\; \inf\Big\{\sum_{k=1}^\infty \mu_0(A_k) : A_k \in \mathcal{A},\, A \subseteq \bigcup_k A_k\Big\}.$$

We then let $\mathcal{M}$ denote the $\sigma$-algebra of $\mu^*$-measurable sets, and we know by Carathéodory's Theorem that $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure. Our next goal is to show that such a measure is "well-behaved."

**Theorem 42.** Given a premeasure $\mu_0$ and associated outer measure $\mu^*$, the following statements hold.

(a) $\mu^*|_{\mathcal{A}} = \mu_0$, i.e., $\mu^*(A) = \mu_0(A)$ for every $A \in \mathcal{A}$.

(b) $\mathcal{A} \subseteq \mathcal{M}$, i.e., every set in $\mathcal{A}$ is $\mu^*$-measurable. Consequently, $\mu(A) = \mu_0(A)$ for every $A \in \mathcal{A}$.

(c) If $\nu$ is any measure on $\mathcal{M}$ such that $\nu|_{\mathcal{A}} = \mu_0$, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality holding if $\mu(E) < \infty$. Furthermore, if $\mu_0$ is $\sigma$-finite, then $\nu = \mu$.

*Proof.* (a) Suppose that $A \in \mathcal{A}$. Then $\{A\}$ is one covering of $A$ by sets from $\mathcal{A}$, so we have

$$\mu^*(A) \;\leq\; \mu_0(A).$$

On the other hand, suppose that $\{A_k\}_k$ is any countable collection of sets in $\mathcal{A}$ that covers $A$. Disjointize these sets by defining

$$B_1 \;=\; A \cap A_1 \qquad \text{and} \qquad B_n \;=\; A \cap \Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big), \quad n > 1.$$

Then $B_1, B_2, \dots \in \mathcal{A}$ and $\cup B_n = A \in \mathcal{A}$, so by the definition of premeasure and the fact that $\mu_0$ is monotonic we have

$$\mu_0(A) \;=\; \sum_{n=1}^\infty \mu_0(B_n) \;\leq\; \sum_{n=1}^\infty \mu_0(A_n).$$

Since this is true for every covering of $A$, we conclude that

$$\mu_0(A) \;\leq\; \mu^*(A).$$

Thus $\mu_0$ and $\mu^*$ agree on $\mathcal{A}$.

(b) Suppose that $A \in \mathcal{A}$ and that $E \subseteq X$. If we fix any $\varepsilon > 0$, then there exists a countable covering $\{A_k\}_k$ of $E$ by sets $A_k \in \mathcal{A}$ such that

$$\sum_{k=1}^{\infty} \mu_0(A_k) \ \le\ \mu^*(E) + \varepsilon.$$

Hence,

$$\begin{aligned}
\mu^*(E) &\le \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}) && \text{(subadditivity)}\\
&\le \mu^*\Big(\Big(\bigcup_k A_k\Big) \cap A\Big) + \mu^*\Big(\Big(\bigcup_k A_k\Big) \cap A^{\mathrm{C}}\Big) && \text{(monotonicity)}\\
&= \mu^*\Big(\bigcup_k (A_k \cap A)\Big) + \mu^*\Big(\bigcup_k (A_k \cap A^{\mathrm{C}})\Big)\\
&\le \sum_{k=1}^{\infty} \mu^*(A_k \cap A) + \sum_{k=1}^{\infty} \mu^*(A_k \cap A^{\mathrm{C}}) && \text{(subadditivity)}\\
&= \sum_{k=1}^{\infty} \Big(\mu_0(A_k \cap A) + \mu_0(A_k \cap A^{\mathrm{C}})\Big) && \text{(part (a))}\\
&\le \sum_{k=1}^{\infty} \mu_0(A_k) && \text{(finite additivity on } \mathcal{A})\\
&\le \mu^*(E) + \varepsilon.
\end{aligned}$$

Since this is true for every $\varepsilon$, we conclude that

$$\mu^*(E) \ =\ \mu^*(E \cap A) + \mu^*(E \cap A^{\mathrm{C}}).$$

Hence $A$ is $\mu^*$-measurable.

(c) Suppose that $\nu$ is any measure on $\mathcal{M}$ that extends $\mu_0$. Suppose that $E \in \mathcal{A}$ is given. If $\{A_k\}_k$ is any countable cover of $E$ by sets $A_k \in \mathcal{A}$, then we have

$$\nu(E) \ \le\ \sum_{k=1}^{\infty} \nu(A_k) \ =\ \sum_{k=1}^{\infty} \mu_0(A_k).$$

Since this is true for every covering, we conclude that

$$\nu(E) \ \le\ \mu^*(E) \ =\ \mu(E).$$

Suppose now that $E \in \mathcal{M}$ and $\mu(E) < \infty$. Given $\varepsilon > 0$, we can find $A_k \in \mathcal{A}$ such that $\cup A_k \supseteq E$ and

$$\sum_k \mu_0(A_k) \ \le\ \mu^*(E) + \varepsilon.$$

Set $A = \cup A_k$. Then

$$\begin{aligned}
\mu(A) &\le \sum_k \mu(A_k) && \text{(subadditivity)} \\
&= \sum_k \mu_0(A_k) && \text{(part (a))} \\
&\le \mu^*(E) + \varepsilon \\
&= \mu(E) + \varepsilon && \text{(since } E \in \mathcal{M}).
\end{aligned}$$

Since all quantities are finite, it follows from additivity that

$$\mu(A \setminus A) \;=\; \mu(A) - \mu(E) \;\le\; \varepsilon.$$

Now, since $\mathcal{A}$ is closed under finite unions, we have that $\cup_{k=1}^N A_k \in \mathcal{A}$ for every $N$. Therefore, by continuity from below and the fact that $\mu$ and $\nu$ both extend $\mu_0$, we have that

$$\nu(A) \;=\; \lim_{N\to\infty} \nu\Big(\bigcup_{k=1}^N A_k\Big) \;=\; \lim_{N\to\infty} \mu_0\Big(\bigcup_{k=1}^N A_k\Big) \;=\; \lim_{N\to\infty} \mu\Big(\bigcup_{k=1}^N A_k\Big) \;=\; \mu(A).$$

Hence

$$\mu(E) \;\le\; \mu(A) \;=\; \nu(A) \;=\; \nu(E) + \nu(A\backslash E) \;\le\; \nu(E) + \mu(A\backslash E) \;\le\; \nu(E) + \varepsilon.$$

Since $\varepsilon$ is arbitrary, it follows that $\mu(E) = \nu(E)$.

Finally, suppose that $\mu_0$ is $\sigma$-finite, i.e., we can write $X = \cup A_k$ with $\mu_0(A_k) < \infty$ for each $k$. By applying the disjointization trick, we can assume that the sets $A_k$ are disjoint. Then since each $A_k$ has finite measure, we have for any $E \in \mathcal{M}$ that

$$\nu(E) \;=\; \sum_{k=1}^{\infty} \nu(E \cap A_k) \;=\; \sum_{k=1}^{\infty} \mu(E \cap A_k) \;=\; \mu(E).$$

Hence $\mu = \nu$. □

### 1.4.9 The Cantor–Lebesgue Function

The Cantor–Lebesgue Function is an interesting function that is often useful for constructing counterexamples. Therefore we digress for a moment to introduce this function and to develop some of its properties in the form of exercises.

Consider the two functions $\varphi_1, \varphi_2$ pictured in Figure 3. The function $\varphi_1$ takes the constant value $1/2$ on the interval $(1/3, 2/3)$ that is removed in the first stage of the construction of the Cantor middle-thirds set, and is linear on the remaining intervals. The function $\varphi_2$ also takes the same constant $1/2$ on the interval $(1/3, 2/3)$ but additionally is constant with values $1/4$ and $3/4$ on the two intervals that are removed in the second stage of the construction of the Cantor set. Continue this process, defining $\varphi_3, \varphi_4, \dots$, and prove the following facts.

(a) Each $\varphi_k$ is monotone increasing on $[0, 1]$.

(b) $|\varphi_{k+1}(x) - \varphi_k(x)| < 2^{-k}$ for every $x \in [0, 1]$.

(c) $\varphi(x) = \lim_{k\to\infty} \varphi_k(x)$ converges uniformly on $[0, 1]$.

The function $\varphi$ constructed in this manner is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase*.

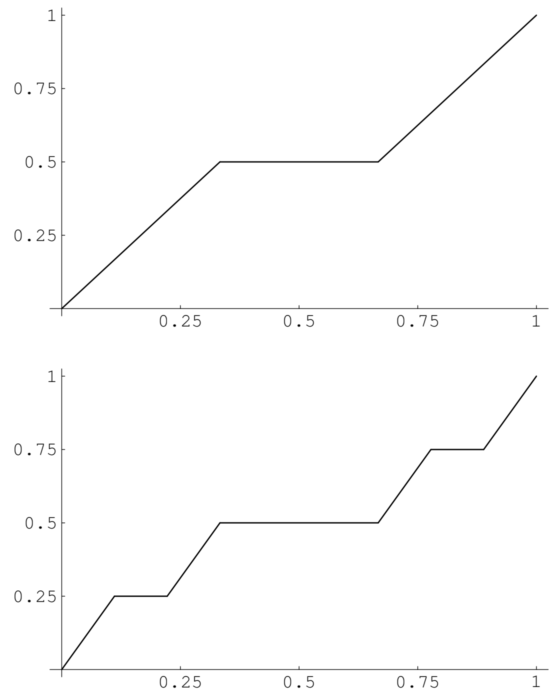


FIGURE 3. First stages in the construction of the Cantor–Lebesgue function.

Prove the following facts about $\varphi$.

(d) $\varphi$ is continuous and monotone increasing on $[0,1]$, but $\varphi$ is not uniformly continuous.

(e) $\varphi$ is differentiable for a.e. $x \in [0,1]$, and $\varphi'(x) = 0$ a.e.

Although we have not yet defined the integral of a function like $\varphi'$ that is only defined almost everywhere, we will later see after we develop the Lebesgue integral that sets of measure zero "don't matter" when dealing with integrals. As a consequence, the Fundamental Theorem of Calculus does *not* apply to $\varphi$:

$$\varphi(1) - \varphi(0) \;\neq\; \int_0^1 \varphi'(x)\,dx.$$

If we extend $\varphi$ to $\mathbb{R}$ by reflecting it about the point $x = 1$, and then extend by zero outside of $[0,2]$, we obtain the continuous function $\varphi$ pictured in Figure 4. It is interesting that it can be shown that $\varphi$ is an example of a *refinable function*, as it satisfies the following *refinement equation*:

$$\varphi(x) \;=\; \frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \frac{1}{2}\varphi(3x-3) + \frac{1}{2}\varphi(3x-4). \tag{7}$$

Thus $\varphi$ equals a finite linear combination of compressed and translated copies of itself, and so exhibits a type of self-similarity. Refinable functions are widely studied and play important roles in wavelet theory and in subdivision schemes in computer-aided graphics.

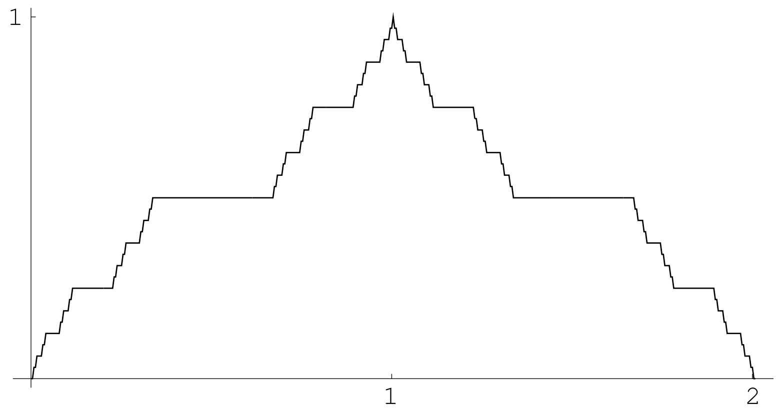


Figure 4. The reflected Devil's staircase (Cantor–Lebesgue function).

The fact that $\varphi$ is refinable yields easy recursive algorithms for plotting $\varphi$ to any desired level of accuracy. For example, since we know the values of $\varphi(k)$ for $k$ integer, we can compute the values $\varphi(k/3)$ for $k \in \mathbb{Z}$ by considering $x = k/3$ in equation (7). Iterating this, we can obtain the values $\varphi(k/3^j)$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Exercise: Plot the Cantor–Lebesgue function.

The Cantor–Lebesgue function is the prototypical example of a singular function.

**Definition 43** (Singular Function)**.** A function $f\colon [a,b] \to \mathbb{C}$ or $f\colon \mathbb{R} \to \mathbb{C}$ is *singular* if $f$ is differentiable at almost every point in its domain and $f' = 0$ a.e.

### 1.4.10 A Nonmeasurable Set

We will give another proof of the existence of subsets of $\mathbb{R}$ that are not Lebesgue measurable. To do this, we need the following theorem.

**Theorem 44.** If $E \subset \mathbb{R}$ is Lebesgue measurable and $|E| > 0$, then

$$E - E \;=\; \{x - y : x, y \in E\}$$

contains an interval around 0.

*Proof.* Given $0 < \varepsilon < 0$, we can find an open $U \supseteq E$ such that $|U| \le (1+\varepsilon)\,|E|$. Since $U$ is an open subset of $\mathbb{R}$ we can write it as a disjoint union of at most countably many intervals, say

$$U \;=\; \bigcup_k (a_k, b_k).$$

Set $E_k = E \cap (a_k, b_k)$. Each $E_k$ is measurable, and $E$ is the disjoint union of the $E_k$. Therefore, we have

$$|U| \;=\; \sum_k (b_k - a_k) \qquad \text{and} \qquad |E| \;=\; \sum_k |E_k|.$$

Since $|U| \le (1+\varepsilon)\,|E|$, we must have

$$b_k - a_k \;\le\; (1+\varepsilon)\,|E_k|$$

for at least one $k$.[1] Fix $d$ with

$$0 \;<\; |d| \;<\; \frac{(1-\varepsilon)\,(b_k-a_k)}{1+\varepsilon}.$$

We have that $E_k \cup (E_k+d) \subseteq (a_k, b_k+d)$ if $d \geq 0$, and $E_k \cup (E_k+d) \subseteq (a_k-d, b_k)$ if $d \leq 0$, In any case,

$$|E_k \cup (E_k+d)| \;\leq\; b_k - a_k + |d|.$$

If $E_k$ and $E_k+d$ were disjoint, then this would imply that

$$b_k - a_k + |d| \;\geq\; |E_k \cap (E_k+d)| \;=\; |E_k| + |E_k+d| \;\geq\; \frac{2(b_k-a_k)}{1+\varepsilon}.$$

Regarranging, we find that

$$|d| \;\geq\; \frac{(1-\varepsilon)\,(b_k-a_k)}{1+\varepsilon},$$

which is a contradiction. Therefore $E_k \cap (E_k+d) \neq \emptyset$ for all $|d|$ small enough. Hence $E_k - E_k$ contains an interval, and therefore $E-E$ does as well. ☐

**Exercise 45.** Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \qquad \Longleftrightarrow \qquad x - y \in \mathbb{Q}.$$

By the Axiom of Choice, there exists a set $N$ that contains exactly one element of each equivalence class for $\sim$. Show that $N-N$ contains no intervals and that $|N|_e > 0$, and conclude that $N$ is not Lebesgue measurable.

## Exercises

Here are some practice exercises on Lebesgue measure on $\mathbb{R}^d$, mostly taken from the text by Wheeden and Zygmund. Be sure to also work the exercises on abstract measure theory from Folland's text.

1. Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the $k$th stage, each interval removed has length $\delta 3^{-k}$, where $0 < \delta < 1$ is fixed. Show that the resulting set has measure $1-\delta$ and contains no intervals.

2. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < \infty$, show that $\limsup E_k$ (and so also $\liminf E_k$) has measure zero.

3. Show that if $E_1$ is a measurable subset of $\mathbb{R}^m$ and $E_2$ is a measurable subset of $\mathbb{R}^n$, then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^{m+n}$, and

   $$|E_1 \times E_2| \;=\; |E_1|\,|E_2|.$$

   Note: Interpret $0 \cdot \infty$ as 0.
   Hint: Use an equivalent characterization of measurability.
   Remark: This is an important fact that we will need to use later.

[1] That is, for at least one $k$, on at least one side.

3. Recall from Theorem 9 that if $E \subseteq \mathbb{R}^d$, then its exterior measure satisfies
$$|E|_e \;=\; \inf\bigl\{|U|_e : U \text{ open}, \, U \supseteq E\bigr\}.$$
Define the *inner Lebesgue measure* of $E$ to be
$$|E|_i \;=\; \sup\bigl\{|F| : F \text{ closed}, F \subseteq E\bigr\}.$$
Prove that $|E|_i \le |E|_e$. Show that if $|E|_e < \infty$, then $E$ is Lebesgue measurable if and only if $|E|_e = |E|_i$. Give an example that shows that this equivalence can fail if $|E|_e = \infty$.

4. Show that if $E \subseteq \mathbb{R}^d$ is measurable and $A \subseteq E$, then $|E| = |A|_i + |E \backslash A|_e$.

5. Give an example of a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ and a measurable set $E \subseteq \mathbb{R}$ such that $f(E)$ is not measurable.
   Hint: Consider the Cantor–Lebesgue function and the preimage of an appropriate nonmeasurable subset of its range.

6. Show that there exist disjoint $E_1, E_2, \dots$ such that
$$\biggl|\bigcup_{k=1}^{\infty} E_k\biggr|_e \;<\; \sum_{k=1}^{\infty} |E_k|_e,$$
with strict inequality.
   Hint: Let $E$ be a nonmeasurable subset of $[0,1]$ whose rational translates are disjoint. Consider the translates of $E$ by all rational $r \in (0,1)$, and use the fact that exterior Lebesgue measure is translation-invariant.

7. Show that there exist $E_1 \supseteq E_2 \supseteq \dots$ such that $|E_k|_e < \infty$ for every $k$ and
$$\biggl|\bigcap_{k=1}^{\infty} E_k\biggr|_e \;<\; \lim_{k\to\infty} |E_k|_e,$$
with strict inequality.

8. Show that if $Z \subseteq \mathbb{R}$ has measure zero, then so does $\{x^2 : x \in Z\}$.